Equivariant cobordism and K-theory

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CONTENTS

INTRODUCTION :

This thesis is mainly a study of the equivariant unitary cobordism U_G^* , G a compact Lie group. It is obtained in Chapter 1 as an example of an equivariant cohomology theory associated to a G-spectrum. In addition to discussing a number of results needed in the rest of the thesis, we set up the U_G^* - spectral sequence and use it to prove that given a compact G x G'- space X such that G acts freely on X, there is a natural multiplicative isomorphism $pr^* : U_{G'}^*(X/_G) \longrightarrow U_{G \times G'}^*(X)$.

In Chapter 2, we define a localized cobordism theory $U_G^*[\Lambda^{-1}]$. Given an n-dimensional G-vector bundle E over a compact G-space X, (G abelian), it is proved that $U_G^*(P(E))[\Lambda^{-1}]$ is freely generated over $U_G^*(X)[\Lambda^{-1}]$ by $1, \rho, \ldots, \rho^{n-1}$ for some natural element $\rho \in U_G^2(P(E))[\Lambda^{-1}]$. This is exploited in Chapter 3 to define $U_G^*[\Lambda^{-1}]$- characteristic classes, to give a partial result about $U_G^*[\Lambda^{-1}]$ - theory of Grassmannians, and prove that (for G abelian)

$$U_G^*(X,A) \underset{U_G^*}{\otimes} RG \cong K_G^*(X,A) \ldots\ldots\ldots\ldots\ldots (*)$$

for X a locally compact G-space and A a closed G-subspace of X.

Chapter 4 deals with the proof of this later result for non-abelian G. For connected G with maximal tonus T a Gysin homomorphism $\Gamma_! : U_T^*(X) \longrightarrow U_G^*(X)$ (X a compact G-space) is defined. Composed with the restriction homomorphism $U_G^*(-) \longrightarrow U_T^*(-)$, the result is multiplication by the bordism class $[G/_T]$.

well-known results of Atiyah, Segal and Singer are then used to verify that the equivariant Todd genus of $^G/_T$ is equal to $1 \in RG$. This leads quite easily to the main theorem (the isomorphism (*) for general G).

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Chapter I: Equivariant Cohomology Theories and Cobordism Theory

§ 1.1 : G-spectra:-

Let G be a compact Lie group. By a G-module, we shall mean finite-dimensional complex representation of G. Let $(\underline{c}_1)$ be the category whose objects are pairs of spaces (X,A), X a locally compact G-space and A a closed G-subspace of X and whose morphisms are proper G-maps of pairs. We shall regard a locally compact G-space X as an object of $(\underline{c}_1)$ by identifying it with $(X,\emptyset)$ ($\emptyset$ is the empty set). Let ab be the category of abelian groups, and homomorphisms. An equivariant cohomology theory $h^*_G = (h^n_G : n \in \mathbb{Z})$ on $(\underline{c}_1)$ is defined to be a sequence of contravariant functors: $h^n_G : (\underline{c}_1) \longrightarrow Ab$ and a sequence of natural transformations $\partial^n : h^n_G \rightarrow (A) \longrightarrow h^{n+1}_G\ (X,A)\ \forall (X,A) \in (\underline{c}_1)$ such that:

(a1) if $f \simeq f^1 : (X,A) \longrightarrow (X^1 A^1)$ ($\simeq$: means G-homotopic by proper G-maps), then $h^n_G\ (f) = h^n_G\ (f^1)$

(a2) if $i : X,\emptyset \longrightarrow X,A$, $j : (A,\emptyset) \longrightarrow (X,\emptyset)$ are the inclusions, then the sequence

$$\cdots\ h^{n-1}_G\ (A) \xrightarrow{\partial^{n-1}} h^n_G\ (X,A) \xrightarrow{h^n_G\ (i)} h^n_G\ (X) \xrightarrow{h^n_G\ (j)} h^n_G\ (A) \xrightarrow{\partial^n} \cdots$$

is exact

(a3) every relative G-homeomorphism $(X,A) \xrightarrow{f} (X^1, A^1)$ of objects of $(\underline{c}_1)$, (i.e. $X/A \rightarrow X^1/A^1$ is a G-homeomorphism) induces an isomorphism $h^*_G\ (f)$, (a4) h_G^* is additive i.e. for any family of spaces (X_α) in $(\underline{c}_1)$, the natural homomorphism:

$$h_G^*(\underset{\alpha}{\cup} X_\alpha) \longrightarrow \prod_\alpha h_G^*(X_\alpha)$$

is an isomorphism.

Notation: If f is a morphism in $(\underline{c}_1)$, we shall write f^* for $h_G^n(f)$ as usual.

As in the ordinary case ([12]), there is a natural 1-1 correspondence between theories (h_G^*) and reduced equivariant cohomology theories $(\tilde{h}_G^*)$ defined on the category of compact G-spaces with base point.

A <u>G-spectrum</u> consists of:

(i) a sequence $(Y_k : k = 0, \pm 1, \pm 2, --)$ of G-spaces with base points

(ii) $\forall$ G-module V, a base point preserving G-map : $V^+ \wedge Y_k \longrightarrow Y_{|V|+k}$ where $|V|$ = dimension of V over C and X^+ denotes the one point compactification of the locally compact G-space X. The diagram:

$$\begin{tikzcd}
(V \oplus W)^+ \wedge Y_k = V^+ \wedge (W^+ \wedge Y_k) \arrow[r] \arrow[dr] & V^+ \wedge Y_{|W| + k} \arrow[d] \\
& Y_{|V| + |W| + k}
\end{tikzcd}$$

is G-homotopy commutative for all V & W.

The importance of G-spectra lies in the fact that each defines an equivariant cohomology theory as was shown by G. Whitehead [24] for G=e. For let $Y_G = \{Y_k\}$ be a G-spectrum and let X be a compact G-space with base point. Let Δ_G be the set of all G-modules, with the partial order relation : $V \leq W$ if $\exists f$ a

G-module V^1 s.t. $V \oplus V^1 = W$. This makes Δ_G into a directed set. Given $V \in \Delta_G$ & $n \in \mathbb{Z}$, define

$$B_n^V = [V^+ \wedge X \;, Y_{|V|+n}]_G^0$$

= the set of G-homotopy classes of based G-maps from $V^+ \wedge X \longrightarrow Y_{|V|+n}$. Given $V \oplus V^1 = W$, define

$$f_V^W : B_n^V \longrightarrow B_n^W$$

by : $f_V^W(V^+ \wedge X \xrightarrow{t} Y_{|V|+n})$ = the composition

$$(V \oplus V^1)^+ \wedge X = V^{1+} \wedge (V^+ \wedge X) \xrightarrow{1 \wedge t} V^{1+} \wedge Y_{|V|+n} \longrightarrow Y_{|W|+n}$$

where the latter map comes from the definition of $\mathcal{Y}_G = \{Y_k\}$. The collection $\{B_n^V, F_V^W\}_{\Delta_G}$ is a direct system and if $W = V \oplus C$, $W^+ = V^+ \wedge S^2$, so $[W^+ \wedge X, ..]_G^0$ is an abelian gp.

<u>Define (1.1.1)</u>:

$$\tilde{\mathcal{Y}}_G^{2n}(X) = \varinjlim [V^+ \wedge X \;, Y_{|V|+n}]_G^0$$ – it is naturally an abelian group. Define :

$$\tilde{\mathcal{Y}}_G^{2n-1}(X) = \tilde{\mathcal{Y}}_G^{2n}(SX).$$

Given a closed G-subspace A of a compact G-space X with base point $x_0 \in A$, define :

$$\tilde{\mathcal{Y}}_G^n(X,A) = \tilde{\mathcal{Y}}_G^n(X \cup_A CA)$$

where CA is the reduced cone on A. As in [11] we can define a map $\sigma_n : \tilde{\mathcal{Y}}_G^n(X) \longrightarrow \tilde{\mathcal{Y}}_G^{n+1}(SX)$ as follows: suppose $n = 2k$ and let $x \in \mathcal{Y}_G^{2k}(X)$ be represented by $f: V^+ \wedge X \longrightarrow Y_{|V|+k}$.

Assign to x the element $\sigma(x)$ represented by the composition:

$$V^+ \wedge S\,(SX) = S^2 \wedge (V^+ \wedge X) \xrightarrow{1 \wedge f} S^2 \wedge Y_{|V|+k} \longrightarrow Y_{|V|+k+1}$$

where the last map comes from $\mathcal{Y}_G$. From the definitions, it is immediate that σ is an isomorphism.

Proposition 1.1.2:

$\tilde{\mathcal{Y}}_G^* = \{ \tilde{\mathcal{Y}}_G^n, \sigma \}$ is a reduced equivariant cohomology theory on the category of compact G-spaces with base point. Hence there is a corresponding equivariant cohomology theory $\mathcal{Y}_G^*$ defined on the category $(\underline{c}_1)$.

Proof:-

The first part follows by using the method of Dyer [12] (p.9-13). The analogue of lemma 3, namely that

$$V^+ \wedge (CA \cup X) = C(V^+ \wedge A) \cup (V^+ \wedge X)$$

for a closed G-subspace A of a compact G-space X with base point $x_0 \in A$ follows by noticing that each of the spaces in question is obtained from $V^+ \times (I \times A) \cup V^+ \times X$ by making the same identifications. The rest of the proof applies equally well here.

The 2nd part of the Proposition 1.1.2 follows from the first part (Dyer [12]) Q.E.D.

Let us recall that as in the ordinary case, G=e, we define $\mathcal{Y}_G^*$ as follows:

Definition 1.1.3:

Let X be a locally compact G-space and A a closed G-subspace of X. Define $\mathcal{Y}_G^n(X) = \tilde{\mathcal{Y}}_G^n(X^+)$,

$$\mathcal{Y}_G^n(X,A) = \tilde{\mathcal{Y}}_G^n(X^+, A^+) .$$

§ 1.2: Continuity:

Define 1.2.1 :

A family F of pairs of closed G-subspaces of a locally compact G-space X is said to be a filtering family if $(L,B), (L',B') \in F \Rightarrow \exists (L'',B'') \in F$ s.t. $\Rightarrow (L'', B'') \subset (L \cap L', B \cap B')$

Proposition 1.2.2 :

Let $\mathcal{Y}_G^*$ be the equivariant cohomology theory associated to a G-spectrum $\mathcal{Y}_G = \{Y_k\}_{k \in \mathbb{Z}}$. Suppose each Y_k is the union of compact G-subspaces each of which can be embedded in a G-module as an equivariant neighbourhood retract. Then given a filtering family F of closed G-subspaces of a locally compact G-space X, the natural homomorphism.

$$\theta : \underset{\overrightarrow{\mathcal{F}}}{\text{Lim}}\, \mathcal{Y}_G^*(L,B) \longrightarrow \mathcal{Y}_G^*\Big(\bigcap_{(L,B)\in F} L, \bigcap_{(L,B)\in F} B\Big)$$

is an isomorphism. In particular, if F is the family of closed G-neighbourhoods of a closed G-subspace A of X, then :

$$\theta : \underset{\longrightarrow}{\text{Lim}}\, \mathcal{Y}_G^*(F) \xrightarrow{\cong} \mathcal{Y}_G^*(A) .$$

Proof:-

Since $\cap(L^+ \cup_{B^+} CB^+) = \cap L^+ \underset{(\cap B^+)}{\cup} C(\cap B^+)$, we need only show that $\underset{\longrightarrow}{\mathrm{Lim}}\ \tilde{\mathcal{Y}}^*_G(L) \overset{\cong}{\longrightarrow} \tilde{\mathcal{Y}}^*_G(\cap L)$,

when all the L's are compact and have a common base point.

Put $A = \cap L$ and let $x \in \tilde{\mathcal{Y}}^*_G(A)$ be represented by the map $f : V^+ \wedge A \longrightarrow Y_k$. Hence $f(V^+ \wedge A) \subset M$, a G-subspace of Y_k s.t. M can be embedded in a G-module W by an embedding $j : M \longrightarrow W$. In addition $\exists$ a neighbourhood N of M in W and a retraction $r : N \longrightarrow M$. By the Tietze extension theorem, extend the map $jof : V^+ \wedge A \longrightarrow W$ to the whole of $V^+ \wedge X$ thus getting a map $h : V^+ \wedge X \longrightarrow W$. Hence we get a map: $G \times (V^+ \wedge X) \longrightarrow W$ defined by $(g,y) \longrightarrow g(h(g^{-1}(y)))$. Intergrating over G (Pontijagin [19]) we obtain an equivariant map $h_1 : V^+ \wedge X \longrightarrow W$ s.t. $jof = h_1 \mid V^+ \wedge A \longrightarrow W$. By continuity of h_1, $\exists$ a G-neighbourhood T of A s.t. $h_1(V^+ \wedge T) \subset N$. We now have a G-map : $V^+ \wedge T \longrightarrow Y_k$ defined by the composition:

$$V^+ \wedge T \xrightarrow{h_1} N \xrightarrow{r} M \xrightarrow{j} Y_k ,$$

which restricted to $V^+ \wedge A$ is $= f$. Since T contains a closed G - neighbourhood of A (by compactness), it follows that

$$\theta : \underset{\longrightarrow}{\mathrm{Lim}}\ \mathcal{Y}^*_G(L) \longrightarrow \mathcal{Y}^*_G(A)$$

is an epimorphism.

Suppose now $x, y \in \tilde{\mathcal{Y}}^*_G(L)$, represented by:

$$f : W^+ \wedge L \longrightarrow Y_k \quad , \quad g : W^+ \wedge L \longrightarrow Y_k$$

satisfy : $i^*(x) = i^*(y)$

where $i : A \subset L$ is inclusion. This means $\exists$ a G-module V s.t. the composition : $V^+\wedge(W^+\wedge A) \xrightarrow{1\wedge f} V^+\wedge Y_k \xrightarrow{(1)} Y_{|V|+k}$

is G-homotopic to the composition:

$$V^+\wedge(W^+\wedge A) \xrightarrow{1\wedge g} V^+\wedge Y_k \xrightarrow{(1)} Y_{|V|+k}$$

(here (1) is the given map of the spectrum) i.e. there is a G-map $(V^+\wedge W^+\wedge A)\times I \cup_{(V^+\wedge W^+\wedge A)\times\partial I} (V^+\wedge W^+\wedge L)\times dI \xrightarrow{f'} Y_{|V|+k}$.

Now $(V^+\wedge W^+\wedge A)\times I \cup_{(V^+\wedge W^+\wedge A)\times\partial I} (V^+\wedge W^+\wedge L)\times dI$ is a closed G-subspace of $(V^+\wedge W^+\wedge L)\times I$. By the same argument used above, $\exists$ a closed G-neighbourhood N' of

$$(V^+\wedge W^+\wedge A)\times I \cup_{(V^+\wedge W^+\wedge A)\times dI} (V^+\wedge W^+\wedge L)\times dI \quad \text{and a G-map}$$

$$\bar{f}' : N' \longrightarrow Y_{|V|+k}$$

extending $f' : (V^+\wedge W^+\wedge A)\times I \cup_{(V^+\wedge W^+\wedge A)\times dI} (V^+\wedge W^+\wedge L)\times dI \longrightarrow Y_{|V|+k}$.

It is very easy to see that $N' \supset (V^+\wedge W^+\wedge L')\times I$ for some $L'\subset L$.

Therefore $j^* x = j^* y$ where $j : L' \subset L$ is the inclusion

i.e. θ is a monomorphism . Q.E.D.

Remark:

The method above can also be used to prove continuity of K_G -theory (Cf Segal [20]). The only non-trivial point is extending a

G-vector bundle E on A to a closed G-neighbourhood of A in X (all spaces being compact). If A is connected, then E is classified by a G-map

$$f : A \to G_k (V)$$

where $k = \dim E$, and $G_k (V)$ is the Grassmann Manifold of (k - dimḷ subspaces) of a G-module V. $G_k (V)$ can be embedded in a G-module W (Palais [18]) since it is a compact differentiable manifold and it is then a G-neighbourhood retract (the normal bundle to $G_k (V)$ = tubular neighbourhood of $G_k(V)$). Hence one can extend E to a neighbourhood of A. One deals with the general case by using the fact that $S^2 \wedge A$ is connected.

§ 1.3 : The equivariant unitary cobordism theory $U^*_G(-)$:-

1.3.1: The Thorm Spectrum $MU_*(G)$:-

Let $\{V_j : j \in J\}$ be a complete set of irreducible inequivalent representations of the compact Lie group G (J is finite or countable according as G is finite or not). Define: the universal G-module V^∞ by:

$$V^\infty = \lim_{\substack{m \to \infty \\ m \in \mathbb{Z}}} m \sum_{i \in J} V_j$$

Let $\gamma^G_k = (E_k (G) \xrightarrow{\pi} B_k (G))$ be the universal k-dimensional G-vector bundle i.e.

$E_k (G) = \{(K,x) : K$ a k-dimḷ plane through the origin in V^∞, and $x \in K\}$, $B_k (G) = \{K : K$ a k-plane through the origin in $V^\infty\}$ & $\pi : E_k (G) \to B_k (G)$ is projection onto the first factor.

The reason these bundles are called universal is :

Proposition 1.3.1 :

Let X be a compact G-space. The assignment:

$$[f] \longrightarrow [f^{*}(\gamma_k^G)]$$

defines a natural 1-1 correspondence between the set of G-homotopy classes of G-maps from X to $B_k(G)$ and the set of isomorphism classes of k-dim^{l}. G-vector bundles on X.

Proof:

Denote as usual the set of G-homotopy classes of G-maps from X to $B_k(G)$ by $[X, B_k(G)]_G$ and the set of isomorphism classes of k-dim^{l}. G-vector bundles on X by $Vect_k^G(X)$.

By Proposition 1.3 (Segal [20]), if $f' \simeq f : X \to B_k(G)$, then $f^{*}(\gamma_k^G) \cong f'^{*}(\gamma_k^G)$. Hence we have a well-defined map:

$$[X, B_k(G)]_G \longrightarrow Vect_k^G(X)$$

On the other hand, if E is a k-dim^{l}. G-vector bundle on X, there is a G-vector bundle $E^{\perp}$ on X s.t.

$E \oplus E^{\perp}$ is trivial, $E \oplus E^{\perp} = \underline{V} = X \times V$ for some G-module $V \subset V^{\infty}$ (Proposition 2.4 Segal [20]). Hence there is an epimorphism $\psi : X \times V \to E$ inducing a G-map :

$f : X \to G_k(V)$ (the Grassmann manifold of k-dim^{l}. subspaces of V) by assigning to $x \in X$, $f(x) = V/\mathrm{Ker}\,\psi_x$.

Proceeding as in Atiyah ([1] P. 29), we can prove that the homotopy class of the G-map :

$$X \xrightarrow{\bar{f}} B_k(G)$$

given by the composition $X \xrightarrow{f} G_k(V) \subset B_k(G)$, does not depend on V or the epimorphism ψ we choose. So we construct an inverse map $\mathrm{Vect}_k^G(X) \to [X, B_k(G)]_G$ by :

$$[E] \to [\bar{f}] \qquad \text{Q.E.D.}$$

By analogy with the ordinary case, $G=e$, where the $B_k(G)$ are the infinite - dimensional Grassmannians, we can see that $B_k(G)$ is paracompact and thus γ_k^G admits an invariant metric (Milnor [15], Husemoller [14]). Define the <u>universal k-dimensional Thom space</u> $M_k(G)$ by

$$M_k(G) = D(E_k(G))/S(E_k(G)) = \frac{\text{the unit ball bundle of } E_kG}{\text{the unit sphere bundle of } E_kG}$$

w.r.t. the above metric. So $M_k(G)$ has a natural base point. Moreover, $\forall$ G-module V, we have a natural G-homotopy class of maps:

$$V^+ \wedge M_k(G) \longrightarrow M_{|V|+k}(G)$$

namely the one induced by the classifying map of the bundle :

$$V \times E_k(G) \longrightarrow pt. \times B_k(G)$$

(Proposition 1.3.1). $MU_*(G) = \{M_k(G)\}$ defines a G-spectrum, called the equivariant. <u>Thom spectrum</u>.

Hence, by § 1.1, the following definitions make sense.

<u>Definition 1.3.2</u> :

(i) Given acompact G-space X with base point, and a closed G-subspace A of X, containing the base point, define $\tilde{U}_G^{2n}(X) = \varinjlim [V^+ \wedge X, M_{|V|+n}(G)]_G^0$,

$$\tilde{U}_G^{2n-1}(X) = \tilde{U}_G^{2n}(SX) \quad \&$$

$$\tilde{U}_G^n(X,A) = \tilde{U}_G^n(X \underset{A}{\cup} CA) .$$

(ii) Given alocally compact G-space X, and a closed G-subspace A of X, define

$$U_G^n(X) = \tilde{U}_G^n(X^+) \ \& \ U_G^n(X,A) = \tilde{U}_G^n(X^+, A^+)$$

It follows from Proposition 1.1.2

Cor 1.3.3:

$U_G^* = \{U_G^n\}$ is an equivariant cohomology theory on the category ($\underline{c}_1$).

COR 1.3.4:

U_G^* is continuous i.e. $\forall$ filtering family F of closed G-subspaces of a locally compact G-space X, the natural homomorphism:

$$\varinjlim_{\mathcal{F}} U_G^*(L,B) \longrightarrow U_G^*\Big(\bigcap_{(L,B)\in\mathcal{F}} L, \bigcap_{(L,B)\in\mathcal{F}} B\Big)$$

is an isomorphism.

Proof:

By Proposition 1.2.2, it is enough to show that for all k, $M_k(G)$ is the union of compact G-subspaces each of which can be embedded in a G-module as a G-neighbourhood retract. Consider a G-module V. Let $M_k(V)$ be the Thom-space of the standard k-dim^l. G-vector bundle $E_k(V)$ on $G_k(V)$ = the Grassmann manifold of k-dim^l. subspaces of V. $M_k(V)$ can be naturally identified with a subspace of $M_k(G)$ and as V varies over the set of G-modules, ΔG, the union of these subspaces is equal to $M_k(G)$. Since $G_k(V)$ is a compact differentiable G-manifold, the disc bundle $D(E_k(V))$ is a compact differentiable G-manifold. According to Palais [18] we can embed $D(E_k(V)$ in a G-module W by an embedding f say. This in turn induces an embedding f' of $M_k(V)$ in $W \oplus \mathbb{C}$ defined by :

$$f'(x) = ((1-\|x\|) f(x), \|x\|).^{(\dagger)}$$

Let $S(E_k(V))$ be the sphere bundle of $E_k(V)$. The map $f' : \text{int } D = D(E_k(V)) \setminus S(E_k(V)) \longrightarrow W \oplus \mathbb{C}$ assigning to x the element $((1-\|x\|) f(x), \|x\|)$ is an embedding of the G-manifold int D in $W \oplus C$. We can identify the normal bundle to this manifold in $W \oplus \mathbb{C}$ with a neighbourhood N of it in $W \oplus C$. Let N' be a small ball with centre the point $(0,1)$ in $W \oplus \mathbb{C}$. $N \cup N^1$ is a neighbourhood of $f'(M_k(V))$, and N retracts onto $f'(\text{int } D)$ while N^1 retracts onto $(0,1)$. We can now find a closed G-neighbourhood of $f'(M_k(V))$ which is a subspace of $N \cup N^1$, and retracts onto $f'(M_k(V))$. This completes the proof.

Q.E.D.

§1.4 (A): Multiplication and some functorial Properties:-

For every pair of integers $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there is a unique G-homotopy class of maps:

$$M_m(G) \wedge M_n(G) \xrightarrow{(m,n)} M_{m+n}(G)$$

It is the one induced by a classifying map of the bundle $\gamma_m^G \times \gamma_n^G$. This in turn induces a multiplication on $\tilde{U}_G^*(X)$ $\forall$ compact G-space X with base point defined as follows: Suppose $x, y \in \tilde{U}_G^*(X)$ are represented by the G-maps $V^+ \wedge X \xrightarrow{u} M_k(G)$ and $W^+ \wedge X \xrightarrow{v} M_l(G)$. Define $xy \in \tilde{U}_G^*(X)$ to be the element represented by the composition:

(†) Read : $((1-\|x\|) f(x), \|x\|)$.

$$\begin{tikzcd}(V\oplus W)^+\wedge X \arrow[r, "1\wedge\Delta"] & (V\oplus W)^+\wedge X\wedge X = (V^+\wedge X)\wedge(W^+\wedge X) \arrow[r, "u\wedge v"] & M_k(G)\wedge M_l(G) \arrow[d, "(k,l)"] \\ & & M_{k+l}(G)\end{tikzcd}$$

where $\Delta : X \to X \times X$ is the diagonal map. This gives $U_G^*(X) = \tilde{U}_G^*(X^+)$ a ring structure $\forall$ locally compact G-space X. In particular, it makes $U_G^* = U_G^*$ (point) a ring with unit 1 represented by:

$$\mathbb{C}^+ \longrightarrow M_1(G) \qquad\qquad \text{(trivial action on } \mathbb{C}\text{)}$$

induced by the classifying map of the bundle $\mathbb{C} \to pt.$

Let $\alpha : G' \to G$ be a homomorphism of Lie groups. Let X be a compact G-space with base point, Y a compact G'- space with base point and $S : Y \to X$ a map such that

$$s(g'y) = \alpha(g') s(y) \ \forall g' \in G', \ \& \ y \in Y \quad \text{.... (*)}$$

We shall call such a map S an <u>α - map</u> for any given $\alpha : G' \to G$. Let $x \in U_G^*(X)$ be represented by: $V^+\wedge X \xrightarrow{f} M_k(G)$. Regard V as a G' - module via: $g'(v) = \alpha(g') v$. In the same way, we regard γ_k^G as a G'- vector bundle. So we get a unique G'- homotopy class:

$$M_k(G) \xrightarrow{\nabla} M_k(G')$$

Hence we can define a multiplicative homomorphism:

$$R\alpha : \tilde{U}_G^*(X) \longrightarrow \tilde{U}_{G'}^*(Y)$$

by: $R\alpha(x)$ is the element of $\tilde{U}_{G'}^*(Y)$ represented by the composition:

$$V^+\wedge Y \xrightarrow{1\wedge s} V^+\wedge X \xrightarrow{f} M_k(G) \xrightarrow{\nabla} M_k(G').$$

Proposition 1.4.1 :

Let $\alpha : G' \to G$, $\alpha^1 : G'' \to G'$ be homomorphisms of Lie groups. Suppose X is a compact based G-space, Y a compact based G'- space and Z a compact based G'' - space. Suppose $S: Y \to X$ is an α- map , $s' : Z \to Y$ an α' - map. The diagram:

$$\begin{tikzcd} \tilde{U}^*_G(X) \arrow[rr, "R\alpha"] \arrow[dr, "R(\alpha\circ\alpha')"'] & & \tilde{U}^*_{G'}(Y) \arrow[dl, "R\alpha'"] \\ & \tilde{U}^*_{G''}(Z) & \end{tikzcd}$$

is commutative.

The proof is straightforward.

If $\alpha: H \to G$ is the inclusion of a subgroup of G & S= id: $X \to X$, we shall call R <u>the restriction homomorphism</u> and denote it by r.

Let $U^* \xrightarrow{r_1} U^*_G$ be the natural ~~restriction~~ homomorphism induced by the inclusion $\{e\} \to G$.† According to Thom [23] , we can identify U^* with the bordism ring of unitary manifolds $\mathfrak{U}_*$ by a natural identification:

$$i : \mathfrak{U}_* \to U^* .$$

Suppose now X is a compact G-space. Let $U^*_G \longrightarrow U^*_G(X)$ be the natural map induced by the map $X \to$ point. We can regard $U^*_G(X)$ as a $\mathfrak{U}_*$ - module by means of the homomorphism:

$$\mathfrak{U}_* \xrightarrow{i} U^* \xrightarrow{r_1} U^*_G \longrightarrow U^*_G(X)$$

Proposition 1.4.2:

$U^*_G(X)$ is an algebra over $\mathfrak{U}^*$ for all compact G-spaces X.

† Give everything trivial G-action.

Lemma 1.4.3.:

Suppose X is a locally compact trivial G-space. The restriction homomorphism : $U_G^*(X) \xrightarrow{r} U^*(X)$

has a natural right inverse :

$$r_1 : U^*(X) \longrightarrow U_G^*(X)$$

Proof:-

Let $x \in U^*(X)$ be represented by :

$$S^{2k} \wedge X^+ \xrightarrow{f} M_{k+n} = M_{k+n}(e)$$

Define $r_1(x)$ to be the element represented by:

$$S^{2k} \wedge X^+ \xrightarrow{f} M_{k+n} \subset M_{k+n}(G)$$

with trivial G-action on $S^{2k} \wedge X^+$.

If we forget about the G-action on γ_{k+n}^G, we receive a (k+n) - vector bundle. Given a G-module $V \subset V^\infty$ = the universal G-module of G, map the standard bundle on $G_{k+n}(V)$ (after forgetting the action of G) isomorphically onto the standard bundle on $G_{k+n}(C^{|V|}) \subset B_{k+n} = B_{k+n}(e)$.

This defines a bundle map:

$$E_{k+n}(G) \longrightarrow E_{k+n}$$

and thus induces a map:

$M_{k+n}(G) \xrightarrow{\nabla} M_{k+n}$. In terms of this map:

$r(r_1(x))$ is represented by:

$$S^{2k} \wedge X^+ \xrightarrow{f} M_{k+n} \subset M_{k+n}(G) \xrightarrow{\nabla} M_{k+n}$$

By the definition of ∇, we can assume that the composition:

$$M_{k+n} \subset M_{k+n}(G) \xrightarrow{\nabla} M_{k+n}$$

is $= \mathrm{id} : M_{k+n} \longrightarrow M_{k+n}$. Hence $ror_1 = \mathrm{id}$. Q.E.D.

§ 1.4 : (B) The Thom Isomorphism Theorem For $U_G^*(-)$:-

(i) Let $\pi : E \longrightarrow X$ be an n-dimensional G-vector bundle over the compact G-space X. By Proposition 1.3.1, we can assign to E a G-homotopy class of maps $[\epsilon : E \longrightarrow En(G)]$, namely the classifying one. This induces a based G-homotopy class of maps $[\epsilon : E^+ \longrightarrow Mn(G)]$. Denote by t_E the element of $U_G^{2n}(E)$ represented by $[\epsilon : E^+ \longrightarrow Mn(G)]$ and call it <u>the Thom class of E over X</u>

(ii) Suppose $\pi' : F \longrightarrow X$ is an m-dimensional G-vector bundle over X. We can regard $E \oplus F$ as a G-vector bundle over E in the natural way. Let

$$\begin{array}{ccc} E \oplus F & \xrightarrow{P_2} & F \\ P_1 \downarrow & & \downarrow \pi' \\ E & \xrightarrow{\pi} & X \end{array}$$

be the natural bundle map,

P_1, P_2 are the projections onto the first and second factors. P_2 induces a G-map of Thom spaces : $(E \oplus F)^+ \xrightarrow{P2} F^+$. Let $\epsilon' : F^+ \longrightarrow Mm(G)$ be a representative of the Thom class of F over X. Define the Thom class of $E \oplus F$ over F to be the element represented by the composition:

$$(E \oplus F)^+ \xrightarrow{P_2} F^+ \xrightarrow{\epsilon'} Mm(G) .$$

(iii) E is trivial i.e. $E = V \times X$ for some G-module V and X locally compact. Again we can define the Thom class t_E to be the one represented by the natural map:

$$(V \times X)^+ \longrightarrow M_{|V|}(G).$$

In each of the situations (i), (ii) or (iii) we can define a Thom homomorphism. Since it is the same kind of construction in (i), (ii) and (iii), we shall do it only for case (i). The bundle map:

$$\begin{tikzcd} E \arrow[r, "\pi\times 1"] \arrow[d, "\pi"'] & X \times E \arrow[d, "1\times\pi"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

(where Δ is the diagonal map) induces a G-map of Thom spaces $E^+ \xrightarrow{(\pi\wedge 1)} X^+ \wedge E^+$. Let $x \in U_G^{2k}(X)$ be represented by:

$$V^+ \wedge X^+ \xrightarrow{\ f\ } M_{|V|+k}(G)$$

<u>Define</u> $\varphi(x) \in \tilde{U}_G^{2k+2n}(E^+)$ to be the element represented by the composition.

$$\begin{tikzcd} V^+ \wedge E^+ \arrow[r, "1\wedge(\pi\wedge 1)"] & V^+ \wedge (X^+ \wedge E^+) = (V^+ \wedge X^+) \wedge E^+ \arrow[r, "f\wedge\epsilon"] & M_{|V|+k}(G) \wedge M_n(G) \arrow[d, "(|V|+k,n)"] \\ & & M_{|V|+k+n}(G) \end{tikzcd}$$

Similarly $\varphi(x)$ can be defined for $x \in U_G^{2k-1}(X)$. The map:

$$\varphi : U_G^*(X) \longrightarrow \tilde{U}_G^*(E^+)$$

is a homomorphism which we call <u>the Thom homomorphism</u>.

Proposition 1.4.4 (tom Dieck [11])

The Thom homomorphism is transitive i.e. let E be an m-dim ℓ. G-vector bundle over the compact G-space X, F an n- dim ℓ. G-vector bundle over X, then the diagrams:

$$\begin{tikzcd} U_G^*(X) \arrow[rr, "\varphi"] \arrow[dr, "\varphi"'] & & U_G^*(E \oplus F) \\ & U_G^*(E) \arrow[ur, "\varphi"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} U_G^*(E) \arrow[rr, "\varphi"] \arrow[dr, "\varphi"'] & & U_G^*(E \oplus E \oplus F) \\ & U_G^*(E \oplus F) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

are commutative.

Th. 1.4.5: The Thom Isomorphism Theonem (T. tom Dieck) [11] P.21)

Given an n-dim ℓ. G-vector bundle E over a compact G-space X, then the Thom homomorphism

$$U_G^k(X) \longrightarrow \tilde{U}_G^{k+2n}(E^+)$$

is an isomorphism. If E is trivial, it is enough to have X locally compact.

Proof:

Case 1: The trivial case $E = V \times X$:

The classifying map of this bundle is covered by the map

$$V \times X \longrightarrow E_n(G)$$

sending $(v,x) \longrightarrow (V,v)$ $\forall (v,x) \in V \times X$. Hence given $x \in U_G^*(X)$ represented by:

$$W^+ \wedge X^+ \xrightarrow{f} M_{|W|+k}(G)$$

one can check that $\phi(x)$ is represented by the composition:

$$\begin{tikzcd} W^+\wedge(V\times X)^+ = V^+\wedge(W^+\wedge X^+) \arrow[r, "1\wedge f"] & V^+\wedge M_{|W|+k}(G) \arrow[d] \\ & M_{|V|+|W|+k}(G) \end{tikzcd}$$

where the last map comes from the definition of the Thom spectrum. Therefore, by the definition of $U_G^*(-)$, ϕ is a monomorphism. To see that it is an epimorphism, suppose $y \in \tilde{U}_G^{n}(E^+)$ is represented by:

$$W^+\wedge E^+ = W^+\wedge (V\times X)^+ \xrightarrow{g'} M_{|W|+l}(G)$$

Then $y = \phi(x)$ where x is represented by:

$$(W\oplus V)^+\wedge X^+ = W^+\wedge(V^+\wedge X^+) \cong W^+\wedge(V\times X)^+ \xrightarrow{f'} M_{|W|+l}(G)$$

Case 2 : the general case:

By (Segal [20] Proposition 2.4) , $\exists$ a G-vector bundle F over X st. $E\oplus F$ is trivial, say $E\oplus F = \underline{V} = V\times X$.

By proposition 1.4.4 and Case 1 above, we have a commutative diagram (*) :

$$\begin{tikzcd} U_G^k(X) \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \tilde{U}^{k+2|V|}((E\oplus F)^+) \\ & \tilde{U}_G^{k+2n}(E^+) \arrow[u, "\varphi"'] \end{tikzcd} \qquad \cdots\cdots\cdots(*)$$

in which the horizontal arrow is an isomorphism. Hence

$$\varphi : \tilde{U}_G^{k+2n}(E^+) \longrightarrow \tilde{U}_G^{k+2|V|}((E\oplus F)^+)$$

is an epimorphism. By the same reasoning, we have a commutative diagram :

$$\begin{tikzcd}
\tilde{U}_G^{k+2n}(E^+) \arrow[rr, "\phi"] \arrow[dr, "\varphi"'] & & \tilde{U}_G^{k+2n+2|V|}((E \oplus \underline{V})^+) \\
& \tilde{U}_G^{k+2|V|}((E \oplus F)^+) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

and the horizontal arrow is an isomorphism. Therefore, ϕ : $\tilde{U}_G^{k+2n}(E+) \longrightarrow \tilde{U}_G^{k+2|V|}(E \oplus F)^+)$ is a monomorphism. Hence it is an isomorphism. By considering the diagram (*), the result follows. Q.E.D.

1.4: (C) <u>The Isomorphism $F(G,H)$</u> :

Let H be a closed subgroup of G and let X be a compact H-space. Recall that the G-space $G \underset{H}{\times} X$ is the space obtained from the cartesian product $G \times X$ by identifying (g,y) with $(gh, h^{-1}y)$ $\forall h \in H$ with G acting on it via:

$$g_1 \, [g,y] = [g_1 \, g,y] \, .$$

<u>Note</u>:

Suppose Y is a compact G-space. Then we have a natural identification of G-spaces:

$$v : G \underset{H}{\times} (Y \times X) \cong Y \times (G \underset{H}{\times} X) \qquad (1.4.6)$$

(Y on the left is regarded as an H-space by restriction) : it sends $[g, (y,x)] \longrightarrow (gy, [g,x])$, and thus its invese is given by :

$$(y, [g,x]) \longrightarrow [g,(g^{-1}y,x)] \, .$$

There is a canonical homomorphism (T.tom Dieck [11] P.24):

$$F(G,H) \; : \; \tilde{U}^*_G (G \underset{H}{x} X) \longrightarrow U^*_H (X)$$

defined as the composition:

$$U^*_G (G \underset{H}{x} X) \;\cong\; \tilde{U}^*_G (G^+ \underset{H}{\wedge} X^+) \xrightarrow{\; r \;} \tilde{U}^*_H (G^+ \underset{H}{\wedge} X^+) \xrightarrow{\; q^* \;} \tilde{U}^*_H (X^+) = U^*_H(X)$$

where q^* is induced by the map:

$$q : X^+ \longrightarrow G^+ \underset{H}{\wedge} X^+$$

defined by : $x \longrightarrow [1,x]$. tom Dieck ([11] P.24) has shown too that $F(G,H)$ is an isomorphism with invese:

$$F(H,G) \; : \; U^*_H (X) \longrightarrow U^*_G (G \underset{H}{x} X) \; .$$

$F(H,G)$ is defined by the composition:-

$$[V^+ \wedge X^+ \;, M_k (H)]^o_H \xrightarrow{(1)}$$

$$[G^+ \underset{H}{\wedge} (V^+ \wedge X^+) \;, G^+ \underset{H}{\wedge} M_k (H)]^o_G \xrightarrow{(2)}$$

$$[G^+ \underset{H}{\wedge} (V^+ \wedge X^+) \;, \qquad M_k (G)]^o_G \xrightarrow{(3)}$$

$$\tilde{U}^*_G (G^+ \underset{H}{\wedge} (V^+ \wedge X^+) \;\overset{(4)}{\cong}\; U^*_G \; (G \underset{H}{x} X)$$

where:

(1) sends $(f: V^+ \wedge X^+ \longrightarrow M_k (H)) \xrightarrow{\text{to}} 1 \wedge f : G^+ \underset{H}{\wedge} (V^+ \wedge X^+) \longrightarrow G^+ \underset{H}{\wedge} M_k(H)$,

(2) is induced by the bundle map: $G \underset{H}{x} E'_k (H) \longrightarrow E_k (G)$,

(3) comes from the definition of U^*_G , & (4) is the invese of the Thom isomorphism (Th 1.4.2) applied to the G-vector bundle $G \underset{H}{x} (V \times X) \longrightarrow G \underset{H}{x} X$.

Lemma 1.4.7:

Let X be a compact G-space, and H a closed subgroup of G. Let $P_2 : G/_H \times X \to X$ be projection onto the 2nd factor. Then we have a commutative diagram:

$$\begin{tikzcd} U_G^*(X) \arrow[r, "P_2^*"] \arrow[d, "r"'] & U_G^*(G/_H \times X) \arrow[d, "v^*", "\cong"'] \\ U_H^*(X) \arrow[r, "F(H,G)"] & U_G^*(G \times_H X) \end{tikzcd}$$

where the isomorphism $U_G^*(\,{}^G/H \times X) \cong U_G^*(G \times_H X)$

is the one induced by the identification (1.4.6):

$$v : G \times_H X = {}^G/H \times X .$$

Proof:

Since $F(H,G) : U_H^*(X) \to U_G^*(G \times_H X)$ is an isomorphism with inverse $F(G,H) : U_G^*(G \times_H X) \to U_H^*(X)$, it is enough to prove that the diagram :

$$\begin{tikzcd} U_G^*(X) \arrow[r, "P_2^*"] \arrow[d, "r"'] & U_G^*(\,{}^G/_H \times X) \arrow[d, "v^*", "\cong"'] \\ U_H^*(X) & U_G^*(G \times_H X) \arrow[l, "F(G,H)"] \end{tikzcd} \qquad \ldots\ldots\ldots(d_2)$$

is commutative.

Suppose $x \in U_G^*(X)$ is represented by

$V^+ \wedge X^+ \xrightarrow{f} M_k(G)$. Then $P_2^*(x)$ is represented by the composition:

$V^+ \wedge ({}^G/_H \times X)^+ \xrightarrow{1 \wedge P_2} V^+ \wedge X^+ \xrightarrow{f} M_k(G)$. $v^*(P_2^*(x))$ is represented by the composition S_1 :

$$V^+ \wedge (G \times_H X)^+ \xrightarrow{1 \wedge v} V^+ \wedge ({}^G/_H \times X)^+ \xrightarrow{1 \wedge P_2} V^+ \wedge X^+ \xrightarrow{f} M_k(G)$$

S_1 is a G-map. Denote it by S_1' when regarded as an H-map by restriction.

$F(G,H)(v^*(P_2^*(x)))$ is represented by the composition:

$$V^+ \wedge X^+ \xrightarrow{1 \wedge q} V^+ \wedge (G \times_H X)^+ \xrightarrow{S_1'} M_k(G) \xrightarrow{\varkappa} M_k(H)$$

where $\varkappa$ is obtained by regarding γ_k^G as an H-vector bundle by restriction. Now the H-map given by the composition:

$$\begin{array}{l} V^+ \wedge X^+ \xrightarrow{1 \wedge q} V^+ \wedge (G \times_H X)^+ \xrightarrow{1 \wedge v} V^+ \wedge ({}^G/_H \times X)^+ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 1 \wedge P_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V^+ \wedge X^+ \end{array}$$

$= V^+ \wedge X^+ \xrightarrow{id \wedge id} V^+ \wedge X^+$. Therefore,

$F(G,H)\,(v^*(P_2^*(x))) = r(x)$ i.e. the diagram (d_2) is commutative. Q.E.D.

§ 1.5 (a): Spectral Sequences:-

Our aim is to set up the U_G^* - spectral sequence, and give an application of it in §1.5.b. We assume a number of definitions all of which can be found in (Segal [20] [21]).

Let $h_G^* = (h_G^k)$ be an equivariant cohomology theory on the category $(\underline{G}_1)$ (see 1.1). Let $F = (F_\alpha)_{\alpha \in A}$ be a closed finite covering of a locally compact G-space X. Let N_F be the nerve of F - a finite simplicial complex. Let $|N_F|$ be its geometrical realization. Define $W_F = \bigcup_\sigma (F_\sigma \times |\sigma|)$ where σ runs through the finite subsets of A s.t. $\bigcap_{\alpha \in \sigma} F_\alpha \neq \emptyset$. W_F is a closed G-subspace of $X \times |N_F|$. Define

$$w : W_F \to X$$

to be projection onto the first factor - a proper G-map since $|N_F|$ is compact. Now define a filtration of W_F by:

$$W_F^P = \bigcup_{\dim \sigma \leq P} (F_\sigma \times |\sigma|) .$$

Proposition 1.5.1:

The natural homomorphism:

$$w^* : h_G^*(X) \longrightarrow h_G^*(W_F)$$

is an isomorphism.

Proof:

By the method of Segal (Atiyah-Segal [4] Proposition 3.1.3). ∎

In the manner of Cartan-Eilenberg ([10] P.333), we associate to the filtration:

$$W_F \supset \dots \supset W_F^P \supset \dots \supset W_F^o$$

a spectral sequence whose $E_2^{P,q}$ term is the p th cohomology of the complex

$$h_G^q(W^o) \to h_G^{q+1}(W_F^1, W_F^o) \to \cdots \to h_G^{P+q}(W_F^P, W_F^{P-1}) \to \cdots$$

& with termination $h_G^*(W_F) \cong h_G^*(X)$ (Proposition 1.5.1)

By a standard argument (Segal [4] §3.2, [21] Proposition 5.1) one can verify

(i) $h_G^{P+q}(W_F^P, W_F^{P-1}) = \prod_{\dim\sigma=p} h_G^q(F\sigma)$ i.e. $E_1^{P,q} = \prod_{\dim\sigma=P} h_G^q(F\sigma)$

(ii) & the differential ∂ : $E_1^{P,q} \longrightarrow E_1^{P+1,q}$

corresponds to the differential of the complex of cochains of N_F with coefficients in the system : $\sigma \longrightarrow h_G^q(F\sigma)$. So:

Proposition 1.5.2:

Let F be a finite closed covering of a locally compact G-space X. There is a spectral sequence $H^P(N_F ; h_G^q(F)) \Longrightarrow h_G^*(X)$ where $h_G^q(F)$ means the coefficient system $\sigma \to h_G^q(F\sigma)$.

Proposition 1.5.3:

Let X be a locally compact G-space, Y a compact G-space on which G acts trivially and $f: X \longrightarrow Y$ a G-map. There is a spectral sequence $H^P(Y ; \mathcal{h}_G^q f) \Longrightarrow h_G^*(X)$ where $\mathcal{h}_G^q f$ is the sheaf on Y associated to the presheaf : $V \longrightarrow h_G^q(f^{-1}(\bar{V}))$. If in addition h_G^* is continuous (see Proposition 1.2.2), $\mathcal{h}_G^q f$ has stalk $h_G^q(f^{-1}y)$ at the point y.

Proof:

By the method of Segal [21] Proposition 5.2. If F is a finite open covering of Y, form the spectral sequence E(F) for the finite closed covering $f^{-1}\bar{F}$ of X (see Proposition 1.5.2). This terminates with $h_G^*(X)$ and begins with the Čech cohomology of the covering F with coeffs in the presheaf

$V \to h^q_G(f^{-1}\bar{V})$. Since Y is compact, the set S of finite open coverings of Y is cofinal in the set of open coverings of Y. Therefore, if we take $\varinjlim \{E(F)\}$, and use the fact that the Cech cohomology of a compact space with coeffs in a presheaf is = its cohomology with coeffs in the associated sheaf (Spanier [22] Chap 6), the result follows. Q.E.D. (for more details see Segal [21]). By Cor. 1.3.4 :

Cor 1.5.4:

Proposition 1.5.3 is true when $h^*_G = U^*_G$. In particular, let X be a compact G-space, $Y = X/G$ & $\pi : X \to X/G$ be projection. There is a spectral sequence $H^p(X/G ; \mathcal{U}^q_G) \Rightarrow U^*_G(X)$ where $\mathcal{U}^q_G$ is the sheaf on X/G associated to the presheaf $V \to U^q_G(\pi^{-1}\bar{V})$. The stalk of $\mathcal{U}^q_G$ at an orbit $Gx = G/G_x$ is $U^q_G(G/G_x)$ where G_x is the stabilizer of x.

§ 1.5 (b) : Free Group Action:

Let G,K be compact Lie groups, and X a compact (G x K)- space s.t. K acts freely on X. Let $pr : X \to X/K$ be the projection. It induces a homomorphism :

$$pr^* : U^*_G(X/K) \longrightarrow U^*_{G\times K}(X)$$

defined by the composition:

$$[V^+ \wedge (X/K)^+ , M_n(G)]^o_G \xrightarrow{(1)}$$

$$[V^+ \wedge X^+ , M_n(G)]^o_G \xrightarrow{(2)}$$

$$[V^+ \wedge X^+ , M_n(G\times K)]^o_{G\times K} \xrightarrow{(3)} U^*_{G\times K}(X),$$

where (1) is induced by $pr : X \to X_{/K}$, (2) is induced by the bundle map $E_n(G) \to E_n(G \times K)$ when $E_n(G)$ is regarded as a $G \times K$ - bundle by giving it trivial K - action, and (3) is the natural map. V is regarded as a $G \times K$- module as follows:

$$(g,k)\, v = gv \quad \forall v \in V,\ (g,k) \in G \times K .$$

Proposition 1.5.5:

Suppose $G \times K$ acts on a compact space X s.t. the action of K on X is free. Then :

$$pr^* : U_G^*(X_{/K}) \to U^*_{G \times K}(X)$$

is an isomorphism.

Proof:

Let $\{{}_1Er\}$ be the spectral sequence $H^p((X_{/K})_{/G} ; \mathcal{U}_G^q) \Rightarrow U_G^*(X_{/K})$ where $\mathcal{U}_G^q$ is the sheaf on $(X_{/K})_{/G}$ associated to the presheaf $V \to U_G^q(\pi^{-1}\bar{V})$ and $\pi : X_{/K} \to (X_{/K})_{/G}$ is projection and let $\{{}_2Es\}$ be the spectral sequence : $H^p(X_{/GxK}; \mathcal{U}^q_{GxK})$ where $\mathcal{U}^q_{GxK}$ is the sheaf on $X/GxK = (X_{/K})_{/G}$ associated to the presheaf $V \to U^q_{GxK}(\pi_1{}^{-1}\bar{V})$ ($\pi_1 : X \to X_{/GxK}$ is projection) (see Cor 1.5.4). Now $pr^* : U_G^*(X_{/K}) \to U^*_{GxK}(X)$ induces a homomorphism of spectral sequences : $\{{}_1E_r\} \to \{{}_2E_r\}$ in the natural way. If we can show that this is an isomorphism at the E_2 - level, it would follow that it is an isomorphism at each E_r - level $r \geq 2$ (Cartan - E. Jenberg [10]) and hence the result would follow .

By Cor 1.5.4, the stalk of $\mathcal{V}_G^q$ at an orbit $G[x]$ is $U_G^q\ ({}^G/_{G'})$ where G' is the stabilizer of $[x] \in {}^X/K$, and the stalk of $\mathcal{V}_{GxK}^q$ at the corresponding orbit $(GxK)x$ is $U_{GxK}^q\ (\ (GxK)/_{G''}\ )$ where $G'' \subset GxK$ is the stabilizer of $x \in X$.

The homomorphism :

$$h : G'' \longrightarrow G'$$

given by projection onto the first factor is an isomorphism. For it is clearly epi. To see that it is mono, suppose $(g,k_1)\ x = x = (g,k_2)\ x$. Since K acts freely on X, then $k_1 = k_2$ and so $G'' \xrightarrow{h} G'$ is an isomorphism. We can also identify $(\ (GxK)\ /_{G''})_{/K}$ and $G/_{G'}$ by the projection map.

Now the diagram :

$$\begin{tikzcd}
U_G^q\ ({}^G/_{G'}) \arrow[r, "Pr^*"] \arrow[d, "F(G,G')"'] & U_{GxK}^q\ (\ (GxK)/_{G''}\ ) \arrow[d, "F(GxK,G'')"] \\
U_{G'}^q \arrow[r, "\cong", "Rh"'] & U_{G''}^q
\end{tikzcd}$$

by Proposition 1.4.1

is commutative and the two vertical arrows are isomorphisms (§ 1.4B). Hence $U_G^q\ (G/G'\ ) \xrightarrow{Pr^*} U_{GxK}^q\ (\ (GxK)/G''\ )$ is an isomorphism. Q.E.D.

§ 1.6 : Multiplicative Equivariant Cohomology Theories:-

As we have demonstrated in the previous section, one of the uses of Proposition 1.5.3 is to enable us to reduce the solution of a number of problems to investigating what happens when only orbits are involved. The proposition we are aiming for now plays a similar role when dealing with multiplicative theories.

Definition 1.6.1:

A multiplicative equivariant cohomology theory on ($\underline{c}_1$) is an equivariant cohomology theory, h_G^*, on ($\underline{c}_1$) such that (i) for each (X,A), (X^1, A^1) in ($\underline{c}_1$), there is a homorphism:

$$\otimes : h_G^i(X,A) \otimes h_G^j(X^1, A^1) \longrightarrow h_G^{i+j}(X\times X^1, X\times A^1 \cup A\times X^1)$$

which is associative, anticommutative, natural under proper G-maps of pairs in ($\underline{c}_1$), has a unit $1 \in h_G^o(pt)$,

(ii) Let $X = X^1$ & let :

$$h_G^i(X,A) \otimes h_G^j(X,A^1) \xrightarrow{U} h_G^{i+j}(X, A\cup A^1)$$

be the induced internal pairing i.e. the composition:

$$h_G^i(X,A) \otimes h_G^j(X,A^1) \xrightarrow{\otimes} h_G^{i+j}(X\times X, X\times A^1 \cup A\times X)$$
$$\xrightarrow{\Delta^*} h^{i+j}(X, A \cup A^1)$$

where $\Delta : X \longrightarrow X\times X$ is the diagonal map. We require that under this internal pairing, $h_G^*(A) \xrightarrow{\delta} h_G^*(X,A)$ is an $h_G^*(X)$ - module homomorphism.

A s an example, we have

Lemma 1.6.2:

U_G^* is a multiplicative equivariant cohomology theory.

Proof:-

Let X, X^1 be compact G-spaces with base point, A, A^1 closed G-subspaces of X, X^1 respectively, and containing the base points.

Let $x \in \tilde{U}_G^{2i}(X)$, $x^1 \in \tilde{U}_G^{2j}(X^1)$ be represented by:

f: $V^+ \wedge X \longrightarrow M_{1V1+i}(G)$, and f': $W^+ \wedge X' \longrightarrow M_{1W1+j}(G)$.

Define $x \otimes x' \in \tilde{U}_G^{2i+2j}(X \wedge X')$ to be the element represented by :

$$(V \oplus W)^+ \wedge X \wedge X^1 = (V^+ \wedge X) \wedge (W^+ \wedge X^1) \xrightarrow{f \wedge f'} M_{1V1+i}(G) \wedge M_{1W1+j}(G)$$

$$\xrightarrow{(1V1+i,\ 1W1+j)} M_{1V\oplus W1+i+j}(G)$$

This defines a multiplication :

$$\otimes : \tilde{U}_G^*(XUCA) \otimes \tilde{U}_G^*(X^1\ UCA) \longrightarrow \tilde{U}_G^*((XUCA) \wedge (X^1\ UCA^1)) \cong$$

$$\cong \tilde{U}_G^*(X \wedge X^1 UC\ (X \wedge A^1\ U\ A \wedge X^1))$$

where the last isomorphism is induced by a natural proper G-homotopy equivalence. Hence given (X,A), $(X', A') \in (\underline{C}_1)$, we receive a homomorphism :

$$\otimes : U_G^*(X,A) \otimes U_G^*(X', A') \longrightarrow U_G^*(X \times X';\ A \times X' U X \times A').$$

It is easy to see that this is associative, anticommutative, natural under morphisms in $(\underline{c}_1)$. In § 1.4, we defined $1 \in U_G^0$ (point). It can also be checked that (ii) holds.

Q.E.D.

<u>Proposition 1.6.3</u>:

Let h_G^* be a multiplicative equivariant cohomology theory. Let $\pi : B \to X$ be a G-map of compact G-spaces and $\gamma_1, \ldots, \gamma_n$ homogeneous elements of $h_G^*(B)$. Let N^* be the free h_G^*-module generated by $y_1, \dots, y_n$ $(h_G^* = h_G^*(\text{point}))$. Suppose every orbit in X has a closed G-neighbourhood F s.t. $\forall$ closed G-subspace F_1 of F , the natural map:

$$h_G^*(F_1) \otimes_{h_G^*} N^* \longrightarrow h_G^*(\pi^{-1} F_1)$$

is an isomorphism. Then for any closed G-subspace Y of X, the map:

$$h_G^*(X,Y) \otimes_{h_G^*} N^* \longrightarrow h_G^*(B, \pi^{-1}(Y))$$

is an isomorphism.

<u>Proof</u>:

By the method of proof of Th. 2.7.8, Atiyah [1] .

Q.E.D.

Chapter 2: The Equivariant Cobordism Theory of Projective Spaces:

§ 2.1 (a) The Conner-Floyd Map μ:

2.1(a): Given a locally compact G-space X and a closed G-subspace A of X, let $K_G^*(X,A)$ be defined as in Segal [22]. We shall assume the following result:

Th 2.1.1 :

Let E be an n-dimensional G-vector bundle over a locally compact G-space X. Then the Thom homomorphism (Segal [20])

$$\varphi_* : K_G^*(X) \longrightarrow K_G^*(E)$$

is an isomorphism.

In contrast to the Thom isomorphism theorem for U_G^*, the proof of which is immediate from the definition and the transitivity of the Thom homomorphism (Th 1.4·5), Th 2.1.1 had to be proved (for E trivial) by using elliptic operators (Atiyah [2]).

By means of Th 2.1.1, we would like to define the Conner-Floyd map:

$$\mu : U_G^*(-) \longrightarrow K_G^*(-)$$

(see C-F [8] P.28 for the case G= e). Given a G-module V, let $M_k(V)$ denote the Thom space of the standard bundle $E_k(V)$ over $G_k(V)$ = the Grassmann Manifold (of k-subspaces) of V. If $V \leq W$ (i.e. $\exists$ V' s.t. $V \oplus V' = W$), then inclusion $E_k(V) \subset E_k(W)$ induces a G-map : $M_k(V) \longrightarrow M_k(W)$ and hence a homomorphism:

$$a_V^W : \tilde{K}_G(M_k(W)) \longrightarrow \tilde{K}_G(M_k(V))$$

Define $\tilde{O}(M_k(G)) = \varinjlim \{M_k(V) ; a_V^W\}$.

Let $\bar{\lambda} E_k(V)$ be the Thom class in $\tilde{K}_G(M_k(V))$ ([20] P.140).

Then $a_V^W(\bar{\lambda} E_k(W)) = \bar{\lambda} E_k(V)$ This defines a natural element $\bar{\lambda}^k$ of $\tilde{O}(M_k(G))$.

Definition:

Let X be a locally compact G-space. Suppose $x \in \tilde{U}_G^{2n}(X^+)$ is represented by $f: V^+ \wedge X^+ \longrightarrow M_{1V1+n}(G)$. By compactness of $V^+ \wedge X^+$, we can find a G-module W s.t. $f(V^+ \wedge X^+) \subset M_{1V1+n}(W)$

Define $\mu(x) \in K(X)$ to be the image of $\bar{\lambda}^{l}$ under the composition:

$$(b_1)\ldots\ldots \tilde{O}(M_l(G)) \xrightarrow{(1)} K_G(M_l(W)) \xrightarrow{f^*} \tilde{K}_G(V^+ \wedge X^+) \underset{\cong}{\overset{(2)}{}} K_G(X)$$

where $l = 1V1+n$, (1) comes from the definition of $\tilde{O}(M_l(G))$, and (2) is the inverse of the Thom isomorphism for the trivial bundle $V \times X \longrightarrow X$ (Atiyah [2]).

Lemma 2.1.1 :

$$\mu : U_G^*(X) \longrightarrow K_G^*(X)$$

is well-defined:

Proof:

(i) To prove μ does not depend on the choice of W:

Suppose W_1 and W_2 are s.t. $f(V^+ \wedge X^+) \subset M_l(W_i)$, $i = 1,2$. Hence $f(V^+ \wedge X^+) \subset M_l(W_1 \oplus W_2)$, and for $i = 1,2$, we have a commutative diagram:

$$\begin{array}{ccc} \tilde{O}(M_l(G)) & \xrightarrow{(1)} & \tilde{K}_G(M_l(W_i)) \\ {\scriptstyle (1)'}\downarrow & & \downarrow{\scriptstyle f^*} \\ \tilde{K}_G(M_l(W_1 \oplus W_2)) & \xrightarrow{f^*} & \tilde{K}_G(V^+ \wedge X^+) \end{array}$$

where the last map is that of the Thom spectrum. Put

$\iota_1 = 1V_1 1$. Using this representative of x, $\mu(x)$ is the image of $\bar{\lambda}^{\iota+\iota_1}$ in the composition:

$$(b_2)\ldots.\ \tilde{O}(M_{\iota+\iota_1}(G)) \xrightarrow{(1)} \tilde{K}_G(M_{\iota*\iota_1}(V_1 \oplus W)) \xrightarrow{f_1^*} \tilde{K}_G((V_1 \oplus V)^+ \wedge X^+) \cong$$
$$\cong K_G(X).$$

By the fact that $\lambda_{E\oplus F} = \lambda_E \cdot \lambda_F$ (Segal [20]), and the multiplicativity of the maps (1), f^* and f_1^*, it follows that the composition (b_1) = the composition (b_2).

Q.E.D.

Lemma 2.1.2:

$$\mu : U_G^*(-) \longrightarrow K_G^*(-)$$

is a natural multiplicative transformation of the two theories:

Proof:

μ is multiplicative because the Thom class (when it $\exists$) in K_G - theory satisfies $\lambda_{E\oplus F} = \lambda_E \cdot \lambda_F$ (Segal [20]). The rest of the statement follows from the definition of μ.

Q.E.D.

2.1(b): The Thom class of a line bundle in K_G - theory:

Let L be a 1-dim $\mathbb{C}$. G-vector bundle over a compact G-space X. Let p(L) be the associated principal U(1) - bundle. Recall that the join p(L) o U(1) consists of all points of the form $(1-t)e + tu$ for $0 \leq t \leq 1$, $e \in p(L)$, $u \in U(1)$. U(1) acts principally on it via:

$$((1-t)e + tu)v = (1-t)ev + t\,uv$$ and G acts on it as follows: $$((1-t)e + tu)g = (1-t)ge + tu.$$

Lemma 2.1.3: $\exists$ a canonical G-homeomorphism:

$$f:\ L^{+} \longrightarrow p(L) \circ U(1)/\,U(1)\ .$$

Proof:

As in the ordinary case, G=e (Conner - F [8] P.19).
We identify the disc bundle of L, D(L) with $p(L) \underset{U(1)}{\times} D^2$ and
$S(L)$ = the sphere bundle of L with $p(L) \underset{U(1)}{\times} S^1$
where D^2 and S^1 are the unit disc and unit sphere in C = the complex plane. Define

$$f' : p(L) \times D^2 \longrightarrow p(L) \circ U(1)$$

by :

$$(e,d) \longrightarrow (1 - |d|)\, e + |d|\, (\bar{d}/_{|d|}) \, .$$

f' is equivariant w.r.t. $U(1)$ - actions. So there is defined a G-map :

$$f : (D(L)\ ,\ S(L)) \longrightarrow (p(L) \circ U(1))/_{U(1)}\ \ ,\ x_0\ )$$

where x_0 is the orbit containing all $1 \cdot u$, $u \in U(1)$.
This induces a G-map:

$$D(L)/_{S(L)} \longrightarrow (p(L) \circ U(1))/_{U(1)}$$

which we denote, too, by f. f is 1 - 1 and onto and so it is a homeomorphism since all spaces are compact Hausdorff .

Q.E.D.

Proposition 2.1.4:

Let L be a 1-dim $\mathbb{C}$ G-vector bundle over a compact G-space X. Identify $X^L \equiv$ the Thom space of L with $(p(L) \circ U(1))/_{U(1)}$ as in 2.1.3. Let L' be the vector bundle associated to the principal $U(1)$ - bundle : $p(L) \circ U(1)$ over X^L .

Then the Thom class of $L = 1 - L'$ in $\widetilde{K}_G(X^L)$.

Proof:

Observe that the definition of the Thom class of an n-dimensional G-vector bundle in K_G - theory as in conner-Floyd ([8]) agrees with the definition of Atiyah [1](P.98-101) because Lemma 2.6.13 in [1] is valid in the equivariant case. Hence proceeding as in C-F [8] Th 4.2, we get the result. Q.E.D.

Cor 2.1.5:

Let V be a G-module and H the Hopf bundle over P(V). Identify $P(V)^H$ with $P(V \oplus 1)$ as in Lemma 2.1.3 (see also [8] P.21). Then $\lambda_H = 1 - H_1$ in $\widetilde{K}_G(P(V \oplus 1))$ where H_1 is the Hopf bundle over $P(V \oplus 1)$.

Proof:

Because $S(V \oplus 1) = S(V) \circ U(1)$, $P(V \oplus 1) = S(V \oplus 1)/U(1) =$

$= (S(V) \circ U(1))/U(1)$. Apply Proposition 2.1.4. Q.E.D.

Cor 2.1.6:

The natural element $\bar{\lambda}^1$ in $\widetilde{O}(M_1(G))$ is represented by $1 - H \in \widetilde{K}_G(P(V \oplus 1))$ $\forall$ G-module V where H is the Hopf bundle over $P(V \oplus 1)$.

Lemma 2.1.7:

The Conner-Floyd map μ sends the Thom class of a G-vector bundle in U_G^* - theory to its Thom class in K_G - theory (when they are defined).

§ 2.2 : Some Remarks on the Structure of $U_G^*(P(V))$:-

Suppose V is a G-module. Let P(V) be the projective space of V, and $H = \{(L, x) : L \in P(V), x \in L\}$ the Hopf bundle over P(V). We can identify the Thom space of H with $P(V \oplus 1)$ as indicated in Lemma 2.1.3 and the proof of Cor 2.1.5. So $M_1(G)$ can be identified with $P(V^\infty \oplus 1)$

Definition 2.2.1:

The natural element $\rho \in U_G^2(P(V))$ is the element represented by:

$$i : P(V)^+ \longrightarrow M_1(G)$$

where $i | P(V)$ is inclusion. Define $\rho_H = \rho \in U_G^2(P(V))$.
This enables us to assign to every G-line bundle L over a compact G-space X an element $\rho_L \in U_G^2(X)$ as follows.
As in the proof of Proposition 1·3·1, $\exists$ a G-module W and a G-map : $f : X \longrightarrow P(W)$ s.t. $f^*(H_1) = L$ where H_1 is the Hopf bundle over P(W). Let ρ be the natural element in $U_G^2(P(W))$. Define $\rho_L = f^*(\rho)$ where

$$f^* : U_G^*(P(W)) \longrightarrow U_G^*(X)$$

is the induced homomorphism. To see that ρ_L does not depend on the choice of W or f, suppose (W', f') is used to define ρ_L. Let $j : P(W) \longrightarrow P(W \oplus W')$, $j' : P(W') \longrightarrow P(W \oplus W')$ be the inclusion and let ρ, ρ', ρ'' be the natural elements in P(W), P(W'), $P(W \oplus W')$ respectively.

Proposition 2.2.3:

(i) $U_G^*(P(\mathbb{C}^n))$ is a free U_G^* - module generated by:

$1, \bar{\rho}_n, \bar{\rho}_n^2, \ldots, \bar{\rho}_n^{n-1}$ where $\bar{\rho}_n \in U_G^2(P(\mathbb{C}^n))$ is the natural element, and subject to the relation:

$$\text{(ii)}\ \bar{\rho}_n^n = 0 .$$

Proof:

(ii) $\bar{\rho}_n = r_1(\rho_n)$ and r_1 is multiplicative. So $\bar{\rho}_n^n = r_1(\rho_n^n) = 0$.

(i) By induction. For $n = 1$, $P(\mathbb{C}^1)$ = point and the result is obviously true.

Suppose $U_G^*(P(\mathbb{C}^k))$ is a free U_G^* - module generated by:

$1, \bar{\rho}_k, \ldots, \bar{\rho}_k^{k-1}$. Let $i: P(\mathbb{C}^k) \subset P(\mathbb{C}^{k+1})$ be the inclusion.

Since $i^*(\bar{\rho}_{k+1}) = \bar{\rho}_k$, the cohomology exact sequence of the pair $(P(\mathbb{C}^{k+1}), P(\mathbb{C}^k))$ gives us a s.e.S.

$$0 \to U_G^*(P(\mathbb{C}^{k+1}), P(\mathbb{C}^k)) \xrightarrow{\pi^*} U_G^*(P(\mathbb{C}^{k+1})) \xrightarrow{i^*} U_G^*(P(\mathbb{C}^k)) \to 0$$

It is a split s.e.s. because by hypothesis $U_G^*(P(\mathbb{C}^k))$ is a free U_G^* - module. Now $U_G^*(P(\mathbb{C}^{k+1}), P(\mathbb{C}^k)) \cong \tilde{U}_G^*(S^{2k})$ and so is a free U_G^* - module generated by the Thom class, $\bar{t}$ say, in $\tilde{U}_G^*(S^{2k})$. $\bar{t} = r_1(t)$ where t is the Thom class in $\tilde{U}^*(S^{2k})$ and $r_1 : \tilde{U}^*(S^{2k}) \to \tilde{U}_G^*(S^{2k})$ is given by L1.4.3.

From (2.2.2), we deduce that ρ_{k+1}^k and $\pi^*(t)$ differ by a unit of U^* i.e. ρ_{k+1}^k is either $\pi^*(t)$ or $-\pi^*(t)$ (Milnor [17]).

Therefore, by naturality,

$$\bar{\rho}_{k+1}^k \text{ is either } \pi^*(\bar{t}) \text{ or } -\pi^*(\bar{t}) .$$

Thus $U_G^*(P(\mathbb{C}^{k+1}))$ is freely generated by:

$$1, \bar{\rho}_{k+1}, \bar{\rho}_{k+1}^2, \ldots, \bar{\rho}_{k+1}^k$$

The induction is complete. Q.E.D.

It turns out, however, that $U_G^*(P(V))$ is not in general a free U_G^* - module on generators:

$$1, \rho, \ldots, \rho^{|V|-1}.$$

T. tom Dieck has a counterexample for $G = \mathbb{Z}_2$.

2.3 Localization:

Let $\Lambda_G = \{ x : x \in U_G^*(\text{point}), \mu(x) = 1 \in RG \}$. Introduce the elements of Λ_G as denominators in U_G^* and let the resulting ring be denoted by $U_G^*[\Lambda_G^{-1}]$. There is a natural homomorphism $U_G^* \xrightarrow{(n)} U_G^*[\Lambda_G^{-1}]$ ([6]). This induces a U_G^*-module structure on $U_G^*[\Lambda_G^{-1}]$.

Now for an object (x,A) in $(\underline{c}_1)$, we define :

$$U_G^*(X,A)[\Lambda_G^{-1}] = U_G^*(X,A) \otimes_{U_G^*} U_G^*[\Lambda_G^{-1}].$$ Given a morphism

f in $(\underline{c}_1)$, define $U_G^*(-)[\Lambda_G^{-1}](f) = U_G^*(f) \otimes \mathrm{id}$ where id : $U_G^*[\Lambda_G^{-1}] \longrightarrow U_G^*[\Lambda_G^{-1}]$ is the identity. In what follows, we shall write Λ instead of Λ_G when there is no fear of confusion.

Proposition 2.3.1:

$U_G^*(-)[\Lambda_G^{-1}]$ is a multiplicative equivariant cohomology theory on the category $(\underline{c}_1)$.

Proof:

The only non-obvious point is exactness of the cohomology sequence. This follows from the fact proved in [6] P.88 that the quotient ring, e.g. $U_G^*[\Lambda^{-1}]$, is flat when regarded as a module over the ring from which it is constructed. Q.E.D.

What we would like to do is to compute the $U_G^*(-)[\Lambda^{-1}]$, theory of projective spaces, $P(V)$ (V a G-module).

Lemma 2.3.2:

Let R be a commutative ring with unit 1, and let S be a subset of R closed under multiplication. Let $R[S^{-1}]$ be the quotient ring obtained from R by taking the elements of S as denominators and $R \xrightarrow{(n)} R[S^{-1}]$ the natural map ([6]) Given a ring homomorphism $h: R \longrightarrow R'$ s.t. $h(s)$ is invertible $\forall\, s \in S$, then $\exists$ a canonical homomorphism:

$$\bar{h} : R[S^{-1}] \longrightarrow R'$$

such that the diagram :

$$\begin{tikzcd} R \arrow[r, "(n)"] \arrow[dr, "h"'] & R[S^{-1}] \arrow[d, "\bar{h}"] \\ & R' \end{tikzcd}$$

is commutative.

Proof:

Define $\bar{h} : R[S^{-1}] \longrightarrow R'$ via : $\bar{h}([x/y]) = h(x)(h(y))^{-1}$.

Q.E.D.

Cor 2.3.3:

There a multiplicative natural transformation of cohomology theories:

$$\bar{\mu} : U_G^*(-)[\Lambda^{-1}] \longrightarrow K_G^*(-).$$

Proof:

Define $\bar{\mu} : U_G^*[\Lambda^{-1}] \longrightarrow RG$ to be the cononical homomorphism given by: $\bar{\mu}([x/y]) = \mu(x)(\mu(y))^{-1} = \mu(x)$. Given (X,A) in $(\underline{c}_1)$, define $\bar{\mu} : U_G^*(X,A)[\Lambda^{-1}] = U_G^*(X,A) \otimes U_G^*[\Lambda^{-1}] \longrightarrow K_G^*(X,A)$ to be $\mu \otimes \bar{\mu}: U_G^*(X,A) \otimes_{U_G^*} U_G^*[\Lambda^{-1}] \longrightarrow K_G^*(X,A)$. Q.E.D.

<u>Definition 2.3.4</u>: Let V be a G-module .

Define $\tilde{\rho} \in U_G^2(P(V))[\Lambda^{-1}]$ to be the image of the natural element $\rho \in U_G^2(P(V))$ (§ 2.2) under the homomorphism:

$$U_G^*(P(V)) \longrightarrow U_G^*(P(V)) \otimes_{U_G^*} U_G^*[\Lambda^{-1}] \text{ sending :}$$

$x \longrightarrow x \otimes 1$.

Given a G-line bundle L over a compact G-space X, define $\tilde{\rho}_L$ to be the image of $\rho_L \in U_G^2(X)$ under the homomorphism :

$$U_G^*(X) \longrightarrow U_G^*(X) \otimes_{U_G^*} U_G^*[\Lambda^{-1}]$$

given by : $x \longrightarrow x \otimes 1$.

<u>Lemma 2.3.5</u>:

Given a G-module V, $\tilde{\mu}(\tilde{\rho}) = 1 - H$ in $K_G(P(V))$ where H is the Hopf bundle over P(V) and $\tilde{\mu}: U_G^*(P(V))[\Lambda^{-1}] \longrightarrow K_G^*(P(V))$ is the natural homomorphism.

<u>Proof:</u>

By definition of $\tilde{\mu}$, $\tilde{\mu}(\tilde{\rho}) = \mu(\rho)$. ρ is represented by : $i: P(V)^+ \longrightarrow M_1(G) = P(V^\infty \oplus 1)$ where $i|P(V)$ is inclusion: Hence $\mu(\rho)$ is the image of $\bar{\lambda}^1$ in $\tilde{O}(M_1(G))$ (see § 2.1) under the composition:

$$\tilde{O}(M_1(G)) \xrightarrow{(1)} K_G(P(V \oplus 1)) \xrightarrow{i^*} K_G(P(V)^+) = K_G(P(V))$$

where (1) is the natural map and $i: P(V) \longrightarrow P(V \oplus 1)$ is inclusion. By Cor 2.1.6, (1) sends $\bar{\lambda}^1$ to $1 - H_1$ where H_1 is the Hopf bundle over $P(V \oplus 1)$. Hence $\mu(\rho) = i^*(1 - H_1) = 1 - H$ in $K_G(P(V))$.

Q.E.D.

Cor 2.3.6:

Suppose L is a G-line bundle over a compact G-space X. Then $\bar{\mu}(\bar{\rho}_L) = 1 - L$ in $K_G^*(X)$.

Proof:

By the naturality of $\bar{\mu}$ and $\bar{\rho}_L$, and Lemma 2.3.5. Q.E.D.

§ 2.4 : $U_G^*(-)[\Lambda^{-1}]$ - theory Of Projective Spaces:

We assume the following:

Th 2.4.1:

Let $L_1, \dots, L_n$ be G-line bundles over a compact G-space X and let H be the Hopf bundle over $P(L_1 \oplus L_2 \oplus \dots \oplus L_n)$. Then $K_G^*(P(L_1 \oplus \dots \oplus L_n))$ is a free $K_G^*(X)$ - module generated by 1, 1-H, $(1-H)^2, \dots, (1-H)^{n-1}$. H satisfies the relation $(H - L_1)(H - L_2)\dots(H - L_n) = 0$ and the image of the Thom class Υ in $K_G^*(P_n, P_{n-1})$ under the natural homomorphism $K_G^*(P_n, P_{n-1}) \xrightarrow{j!} K_G^*(P_n)$ is $(H - L_1)(H - L_2)\dots(H - L_{n-1})$, where $P_m = P(L_1 \oplus L_2 \oplus \dots \oplus L_m)$ $(1 \leq m \leq n)$ (P_n/P_{n-1} can be identified with the Thom space of $L_1 \oplus L_2 \oplus \dots L_{n-1}$).

For the proof, see Segal [20] Proposition 3.9 or [4].

Proposition 2.4.2:

Let G be abelian, and V a G-module. Decompose V in the form $V = L_1 \oplus \dots \oplus L_n$ ($\dim L_i = 1$) Let H be the Hopf bundle over P(V). $U_G^*(P(V))[\Lambda^{-1}]$ is freely generated over $U_G^*[\Lambda^{-1}]$ by $1, \bar{\rho}_H, \bar{\rho}_H^2, \dots, \bar{\rho}_H^{n-1}$.

Proof:

By induction. It is obviously true for $n = 1$. Denote $P(L_1 \oplus \ldots \oplus L_m)$ by P_m $(1 \leq m \leq n)$, the Hopf bundle over P_m by H_m, and $\bar{\rho}_{Hm}$ by $\bar{\rho}_m$ Suppose $U_G^*(P_m)[\Lambda^{-1}]$ is freely generated over $U_G^*[\Lambda^{-1}]$ by : $1, \bar{\rho}_m, \ldots, \bar{\rho}_m^{m-1}$. Let $i : Pm \subset Pm-1$ be inclusion. $i^*(\bar{\rho}_{m+1}) = \bar{\rho}_m$ and so the cohomology exact sequence gives us a short exact sequence (s.e.s.) :

$$0 \rightarrow U_G^*(P_{m+1}, P_m)[\Lambda^{-1}] \xrightarrow{j^*} U_G^*(P_{m+1})[\Lambda^{-1}] \xrightarrow{i^*} U_G^*(P_m)[\Lambda^{-1}] \rightarrow 0 .$$

Since $U_G^*(P_m)[\Lambda^{-1}]$ is free over $U_G^*[\Lambda^{-1}]$, it is also split. By the Thom isomorphism theorem (Th 1.4.5), $U_G^*(P_{m+1}, P_m)[\Lambda^{-1}]$ is a free $U_G^*[\Lambda^{-1}]$ - module generated by the Thom class $t = t(L_1 \oplus \ldots \oplus L_m)$ (by this we mean the element $t_m \otimes 1$ in $U_G^*(P_{m+1}, P_m) \otimes_{U_G^*} U_G^*[\Lambda^{-1}]$ where $t_m = t_{L_1 \oplus \ldots \oplus L_m}$ (see § 1.4).

Hence $U_G^*(P_{m+1})[\Lambda^{-1}]$ is freely generated by :

$1, \bar{\rho}_{m+1}, \bar{\rho}_{m+1}^2, \ldots, \bar{\rho}_{m+1}^{m-1}, j^*(t)$. In particular, $\exists$ elements $a_o, a_1, \ldots, a_{m-1}, b \in U_G^*[\Lambda^{-1}]$ such that :

$$\prod_{k=1}^{m} (\bar{\rho}_{m+1} - \bar{\rho}_{L_k}) = \sum a_s \bar{\rho}_{m+1}^{s} + b. j^*(t) \ldots\ldots \quad \text{(i)}$$

If we take $\bar{\mu}$ of each side of (i), we get

$$(-)^m \prod_{k=1}^{m} (H_{m+1} - L_k) = \sum (-)^s \bar{\mu}(a_s) \ (H_{m+1}-1)^s + \bar{\mu}(b)\ \bar{\mu} j^*(t),$$

(Cor 2.3.5). By considering the commutative diagram:

$$\begin{tikzcd} U_G^*(P_{m+1}, P_m)[\Lambda^{-1}] \arrow[r, "j^*"] \arrow[d, "\bar{\mu}'"] & U_G^*(P_{m+1})[\Lambda^{-1}] \arrow[d, "\bar{\mu}"] \\ K_G^*(P_{m+1}, P_m) \arrow[r, "j!"] & K_G^*(P_{m+1}) \end{tikzcd}$$

and Th 2.4.1, we deduce : $\bar{\mu} j^*(t) = j!\, \bar{\mu}'(t)$

i.e. $\bar{\mu} j^*(t) = j!(\tau)$ (Lemma 2.1.7)

$$= \prod_{k=1}^{m} (H_{m+1} - L_k) \qquad \text{(Th 2.4.1). Thus}$$

$$(-)^m \prod_{k=1}^{m} (H_{m+1} - L_k) = \sum (-)^s \bar{\mu}(a_s)(H_{m+1} - 1)^s + \bar{\mu}(b) \prod_{k=1}^{m} (H_{m+1} - L_k).$$

Since $1, 1-H_{m+1}, (1-H_{m+1})^2, \ldots, (1-H_{m+1})^{m-1}$ & $\prod_{k=1}^{m} (H_{m+1} - L_k)$

generate $K_G^*(P_{m+1})$ freely over RG (Th 2.4.1), it follows that

$\bar{\mu}(b) = 1$ or -1. Hence b is an invertible element of

$U_G^*[\Lambda^{-1}]$ and $U_G^*(P_{m+1})[\Lambda^{-1}]$ is a free $U_G^*[\Lambda^{-1}]$-module

generated by : $1, \bar{\rho}_{m+1}, \bar{\rho}^2_{m+1}, \ldots, \bar{\rho}^{m-1}_{m+1}$, & $\prod_{k=1}^{m} (\bar{\rho}_{m+1} - \bar{\rho}_{L_k})$

i.e. we may also take as generators :

$$1, \bar{\rho}_{m+1}, \bar{\rho}_{m+1}, \ldots, \bar{\rho}^m_{m+1}. \qquad \text{Q.E.D.}$$

<u>Cor 2.4.3:</u>

Let $E = X \times V$ be a trivial G-vector bundle over the compact G-space X (G abelian). Let H_o be the Hopf bundle over P(E).
Then $U_G^*(P(E))[\Lambda^{-1}]$ is a free $U_G^*(X)[\Lambda^{-1}]$-module generated by
$1, \bar{\rho}_{H_o}, \bar{\rho}^2_{H_o}, \ldots, \bar{\rho}^{|V|-1}_{H_o}$.

Proof:-

Again by induction. Put $V = L_1 \oplus \dots \oplus L_n$ ($\dim L_i = 1$). Let $\pi_m : X \times P_m \longrightarrow P_m$ be projection onto the 2nd factor. Suppose the~~ir~~ result is true for $X \times P_m$. We have a commutative diagram:

$$\begin{tikzcd}
U_G^*(X \times P_{m+1}, X \times P_m)[\Lambda^{-1}] \arrow[r, "j^*"] & U_G^*(X \times P_{m+1})[\Lambda^{-1}] \\
U_G^*(P_{m+1}, P_m)[\Lambda^{-1}] \arrow[u, "\pi^*"] \arrow[r, "j_2^*"] & U_G^*(P_{m+1})[\Lambda^{-1}] \arrow[u, "\pi^*_{m+1}"']
\end{tikzcd}$$

With the notation used in the proof of the previous proposition, we have shown that:

$$\prod_{k=1}^{m} (\bar\rho_{m+1} - \bar\rho_{L_k}) = \sum a_s \bar\rho^{\,s}_{m+1} + b.j_2^*(t)$$

where b is invertible in $U_G^*[\Lambda^{-1}]$. Now $\pi^*_{m+1}(\bar\rho_{m+1}) = \bar{\bar\rho}_{m+1}$ = the natural element in $U_G^2(X \times P_{m+1})[\Lambda^{-1}]$. On the other hand, $\pi^*(t) = \bar t$ = the Thom class in $U_G^*(X \times P_{m+1}, X \times P_m)[\Lambda^{-1}]$.

Hence $\prod_{k=1}^{m} (\bar{\bar\rho}_{m+1} - \rho_{L_k}) = \sum a_s \bar{\bar\rho}^{\,s}_{m+1} + b\, j^*(\bar t)$.

By the same argument as in the proof of Proposition 2.4.2, we conclude that $U_G^*(X \times P_{m+1})[\Lambda^{-1}]$ is freely generated by:

$1, \bar{\bar\rho}_{m+1}, \bar{\bar\rho}^{\,2}_{m+1}, \dots, \bar{\bar\rho}^{\,m}_{m+1}$. Q.E.D.

Proposition 2.4.4:

Let G be abelian. Suppose E is an n-dimensional G-vector bundle over the compact G-space X. Then $U_G^*(P(E))[\Lambda^{-1}]$ is a free

$U_G^*(X)$ - module generated by : $1, \bar{\rho}, ..., \bar{\rho}^{n-1}$ where $\rho \in U_G^2(P(E))[\Lambda^{-1}]$ is the natural element .

Proof:-

Given $x \in X$, let G_x be the stabilizer. The fibre E_x is a G_x-module. Because G is abelian, one can regard it as the restriction of a G-module. Then $E \mid Gx = G \times_{G_x} E = (G/G_x) \times E_x$. So E and X x E are isomorphic on the orbit Gx and hence on a closed neighbourhood F of it. Let F_o be a G-neighbourhood of Gx and a closed subspace of F. By Cor 2.4.3, $U_G^*(P(F_o \times E_x))[\Lambda^{-1}]$ is a free $U_G^*(F_o)[\Lambda^{-1}]$ - module generated by $1, \bar{\bar{\rho}}, ..., \bar{\bar{\rho}}^{n-1}$ where $\bar{\bar{\rho}} \in U_G^2(P(F_o \times E_x))[\Lambda^{-1}]$ is the natural element. The isomorphism $E \mid F_o \cong F_o \times E_x$ induces an isomorphism of the projective bundles $\beta : P(E \mid F_o) \cong P(F_o \times E_x)$, and of the Hopf bundles over them. Therefore, the natural elements $\bar{\rho} \in U_G^2(P(E \mid F_o))[\Lambda^{-1}]$ and $\bar{\bar{\rho}} \in U_G^2(P(F_o \times E_x))[\Lambda^{-1}]$ correspond to one another under β i.e. $U_G^*(P(E \mid F_o))[\Lambda^{-1}]$ is a free $U_G^*(F_o)[\Lambda^{-1}]$- module generated by $1, \bar{\rho}, ..., \bar{\rho}^{n-1}$.

By Proposition 2.3.1 and Proposition 1.6.3, the result now follows:

Q.E.D.

Chapter 3 : Applications

3.1 : Characteristic Classes:-

Let $h_G^* = \{h_G^n\}$ be a multiplicative equivariant cohomology theory on (c_1). Suppose that $\forall$ G-module V, $\exists$ a natural element $\sigma \in h_G^2(P(V))$ such that if $i: P(V_1) \longrightarrow P(V_2)$ is inclusion, then $i^*(\sigma_2)$ equals σ_1 where σ_j equals the natural element $\sigma \in h_G^2(P(V_j))$. Assign to every G-line bundle L over a compact G-space X, an element $c_1(L) \in h_G^2(X)$ by the requirements : (a) if H is the Hopf bundle over P(V), $c_1(H) = \sigma \in h_G^2(P(V))$ and (b) if $f: X \longrightarrow P(W)$ is a classifying map for L, and $f^*: h_G^2(P(W)) \longrightarrow h_G^2(X)$ the induced homomorphism, then $c_1(L) = f^*(c_1(H))$ where H is the Hopf bundle over P(W). That $c_1(L)$ does not depend on the choice of f follows as in Definition 2.2.1. If $L' = f^*(L)$, $c_1(L') = f^*(c_1(L))$.

Proposition 3.1.1:

Let G be abelian. Suppose that $\forall$ compact G-space X and $\forall$ G-module V, $h_G^*(P(V \times X))$ is a free $h^*_G(X)$ - module generated by : $1, \sigma, \sigma^2, \ldots, \sigma^{|V|-1}$

where σ = the natural element in $h_G^2(P(V \times X))$. Then we can assign to every n-dimensional G-vector bundle E over a compact G-space X a Chern class:

$$c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_n(E) \quad , \quad c_i(E) \in h_G^{2i}(X).$$

Moreover, if $E' = f^*(E)$, $c(E') = f^*(c(E))$.

Proof:

If $|E| = 1$, define $c_1(E)$ as above .

Suppose $n = |E| > 1$. Let $\sigma = c_1(H)$ where H is the Hopf bundle over P (E). Because G is abelian, E is locally decomposable ([20] Proposition 3.7). By Proposition 1.6.3, it follows as in the proof of Proposition 2.4.4, that $h_G^*(P(E))$ is a free $h_G^*(X)$ - module generated by :

$1, \sigma, \sigma^2, \ldots, \sigma^{n-1}$. Define $c_1(E), \ldots, c_n(E)$ to be the unique elements in $h_G^*(X)$ satisfying :

$$(\underline{3.1.2}) \qquad \sum_{i=0}^{n} (-)^i \, c_i(E) \, \sigma^{n-1} = 0 \, . \qquad\qquad \text{Q.E.D.}$$

Suppose that h_G^* satisfies the additional property :-

(*) given based G-maps of compact based G-spaces $f: X \to X^1$, $g: Y \to Y^1$ and ~~at~~ $a \in \tilde{h}_G^i(X')$, $b \in \tilde{h}^j(Y')$, then $(f \wedge g)^*(a \otimes b) = f^* a \otimes g^*(b)$.

In that case,

Proposition 3.1.3:

The Chern classes defined by (3.1.2) satisfy the Whitney Product Theorem i.e. given an m-dimensional G-vector bundle E_1 and an n-dimensional G-vector bundle E_2 over the compact G-space X , then $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$.

Proof:

Put an invariant metric on $E_1 \oplus E_2$. Then $P(E_1 \oplus E_2) = S(E_1 \oplus E_2)/U(1)$ where $S(E_1 \oplus E_2) = \{(e_1, e_2) \mid \|e_1\|^2 + \|e_2\|^2 = 1\}$.

Let A & B be the closed subspaces of $P(E_1 \oplus E_2)$ defined by :

$A = \{[(e_1, e_2)]: \|e_2\|^2 \leq \frac{1}{2}\}$, $B = \{[e_1, e_2] : \|e_1\|^2 \leq \frac{1}{2}\}$.

Then A properly deformation retracts onto $P(E_1)$, B properly deformation retracts onto $P(E_2)$, and $P(E_1 \oplus E_2) = A \cup B$. Let σ be the natural element in $h_G^2(P(E_1 \oplus E_2))$. Then

$x = \sum_{i=o}^{m} (-)^i \; c_i(E_1)\, \sigma^{m-i}$ restricts to 0 in $h_G^*(A)$. Let

$j_1 : (A \cup B, \emptyset) \longrightarrow (A \cup B, A)$ be the inclusion. $\exists x_1 \in h_G^*(A \cup B, A)$ such that $j_1^*(x_1) = x$. Similarly, if $y = \sum_{k=o}^{n} (-)^k \; c_k(E_2)\, \sigma^{n-k}$

and $j_2 : (A \cup B, \emptyset) \longrightarrow (A \cup B, B)$ is the inclusion, $\exists\, y_1 \in h_G^*(AUB, B)$ such that $j_2^*(y_1) = y$. The element $x_1 \otimes y_1$ which

$\in h_G^*((AUB) \times (AUB), A \times (AUB) \cup (AUB) \times B)$ then has

$\beta^*(x_1 \otimes y_1) = x \otimes y$ where

$\beta : (AUB) \times (AUB) \subset ((AUB) \times (AUB), A \times (AUB) \cup (AUB) \times B)$

(because multiplication satisfies the property (*)) .

Therefore, $x \cdot y = 0$ in $h_G^*(AUB)$ since $h_G^*(Y,Y) = 0$ for any (Y,Y) i.e.

$$\left(\sum_{i=o}^{m} (-)^i \; c_i(E_1)\, \sigma^{m-i}\right) \left(\sum_{k=o}^{n} (-)^k \; c_k(E_2)\, \sigma^{n-k}\right)$$

$= 0$ in $h_G^*(P(E_1 \oplus E_2))$. Also

$$\sum_{i=o}^{m+n} (-)^i \; c_i(E_1 \oplus E_2)\, \sigma^{m+n-i} = 0 .$$ Hence by

Proposition 3.1.1., $c_k(E_1 \oplus E_2) = \sum_{i+i=k} c_i(E_1) \cdot c_j(E_2)$. Q.E.D.

Remark:

If h_G^* admits Chern classes satisfying : $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$, then for all compact connected G-spaces X, we can define an additive homorphism:

$$\underline{c}_1 : K_G(X) \longrightarrow h_G^2(X)$$

by : $[E] \longrightarrow c_1(E)$.

As examples of equivariant cohomology theories admitting Chern classes:

Cor 3.1.4:

Let G be abelian. Given an n-dimensional G-vector bundle E over a compact G-space X, let $\sigma = \bar{\rho}_H$ (Def.2.3.4) where H is the Hopf bundle over P(E). We can assign to E a class $cf(E) = 1 + cf_1(E) + \ldots + cf_n(E)$, $cf_i(E) \in U_G^{2i}(X)[\Lambda^{-1}]$, given by:

$$\sum_{i=0}^{n} (-)^i \, cf_i(E) \, \bar{\rho}_H^{\,n-i} = 0 \quad . \qquad (*)$$

cf is natural w.r.t. bundle maps and $cf(E_1 \oplus E_2) = cf(E_1) \cdot cf(E_2)$

Proof:

From propositions 2.4.4, 3.1.1, 3.1.3. Q.E.D.

Cor 3.1.5:

Let E & X be as above. We can assign to E a class $\overline{cf}(E) = 1 + \overline{cf}_1(E_1) + \ldots + \overline{cf}_n(E)$, $\overline{cf}_i \in K_G^{2i}(X)$, defined by

$$\sum_{i=0}^{n} (-)^i \, \overline{cf}_i(E) \, (1-H)^{n-i} = 0 \quad . \qquad (**)$$

Proposition 3.1.6:

Let $\bar{\mu}' : U_G^*(X)[\Lambda^{-1}] \longrightarrow K_G^*(X)$ be the canonical map. Then

$$\overline{cf}(E) = \bar{\mu}(cf(E)) \ .$$

Proof:

Let $\bar{\mu}' : U_G^*(P(E))[\Lambda^{-1}] \to K_G^*(P(E))$. By Cor 2.3.6, $\bar{\mu}'(\bar{\rho}_H) = 1 - H$. Taking $\bar{\mu}'$ of (*) and using the fact that $\bar{\mu}$ is natural and multiplicative, we get:

$$\sum_{i=o}^{n} (-)^i \ \bar{\mu}(cf_i(E)) \ (1-H)^{n-i} = 0 \text{ . Comparing}$$

with (**), $\overline{cf}_i(E) = \bar{\mu}(cf_i(E))$. Q.E.D.

§ 3.2 : An embedding of $K_G(X,A)$ in $U_G^o(X,A)[\Lambda^{-1}]$:

From the results of § 3.1, we can define, $\forall$ compact connected G-space X, a homomorphism :

$$\underline{c}_1 : K_G(X) \longrightarrow U_G^2(X)[\Lambda^{-1}]$$

by: $\underline{c}_1([E] - [E^1]) = cf_1(E) - cf_1(E')$. We assume that G is abelian.

Lemma 3.2.1:

Let X be a compact connected G-space with base point (G abelian).

The composition:

$$\tilde{K}_G(X) \xrightarrow{\underline{c}_1} \tilde{U}_G^2(X)[\Lambda^{-1}] \xrightarrow{\bar{\mu}} \tilde{K}_G(X)$$

is $= -\mathrm{id} : \tilde{K}_G(X) \longrightarrow \tilde{K}_G(X)$.

Proof:

By induction on the dimension of E over X. If L is a G-line bundle, then $\bar{\mu}(cf_1(L)) = \overline{cf}_1(L) = 1 - L$ (Proposition 3.1.6) i.e. $\bar{\mu}(cf_1(L-1)) = -(L-1)$.

Suppose that $\forall$ k-dimensional G-vector bundle E over X,

$\overline{cf}_1\ (E - k) = -\ (E - k)\ (\overline{cf} = \mu \circ cf)$. Let E_1 have dimension

k+1. Let $p: P(E_1) \longrightarrow X$ be the projection, and H the Hopf bundle

over $P\ (E_1)$. $p^*\ (E_1) = H \oplus E$ and by Th 2.4.1 $p^* : K_G\ (X) \longrightarrow K_G\ (P(E_1))$

is injective. Hence $p^*\ \overline{cf}_1\ (E_1 - \overline{k+1}\ ) = \overline{cf}_1\ (H+E - \overline{k+1})$ *

$= -\ (H + E - \overline{k + 1})$ (by the induction hypothesis) $= -\ p^*(E_1 - \overline{k+1})$

i.e. $\overline{cf}_1\ (E_1 - \overline{k+1}) = -\ (E - \overline{k+1})$.

The induction is complete. Q.E.D.

Given a compact G-space X with base point, form $S^2 \wedge X$. It is a

connected space. So we can define a homomorphism:

$$\underline{c}_o : \tilde{K}_G\ (X) \longrightarrow \tilde{U}^o_G\ (X)\ [\Lambda^{-1}] \qquad \text{(G abelian)}$$

as the composition:

$$\tilde{K}_G\ (X) \cong \tilde{K}^2_G\ (S^2 \wedge X) \cong \tilde{K}_G\ (S^2 \wedge X) \xrightarrow{\underline{c}_1} U^2_G\ (S^2 \wedge X)[\Lambda^{-1}] =$$

$$\cong \tilde{U}^o_G\ (X)[\Lambda^{-1}]$$

where all the isomorphisms are the canonical ones.

<u>Cor 3.2.2</u>:

Let G be abelian. Given a locally compact G-space X and a closed G-subspace A of X, $\exists$ an additive homomorphism

$$\underline{c}_o : K_G\ (X,A) \longrightarrow U^o_G\ (X,A)[\Lambda^{-1}]$$

such that $\bar{\mu} \circ \underline{c}_o = -\ \mathrm{id} : K_G\ (X,A) \longrightarrow K_G\ (X,A)$.

<u>Proof</u>:

Given a compact G-space X with base point, the diagram:

$$\begin{tikzcd}
\tilde{K}_G(X) \arrow[d, "\cong"'] & & \tilde{U}^0_G(X)[\Lambda^{-1}] \arrow[ll, "\bar{\mu}"'] \\
\tilde{K}^2_G(S^2 \wedge X) & & \tilde{U}^2_G(S^2X)[\Lambda^{-1}] \arrow[ll, "\bar{\mu}"'] \arrow[u, "\cong"'] \\
& \tilde{K}_G(S^2 \wedge X) \arrow[ul, "\cong"] \arrow[ur, "c_1"'] &
\end{tikzcd}$$

is commutative. Now apply Lemma 3.2.1. Q.E.D.

§ 3.3 : $U_G^*(-)[\Lambda^{-1}]$ of Grassmannians :

Proposition 3.3.1 :

Let E be an n-dim^l G-vector bundle over a compact G-space X and let F(E) be the flag bundle of E. Let $L_1, \dots, L_n$ be the natural line bundles over F(E). Then $K_G^*(F(E))$ is a free $K_G^*(X)$ - module generated by a finite number of elements of the form $L_1^{k_1} L_1^{k_2} \dots L_n^{k_n}$ where $k_i \in \mathbb{Z}^+ \ \forall i$.

Proof:

If $n=1$, $F(E) = X$ and the proposition is true. Suppose the proposition is true for $n-1$ $(n > 1)$ and let E have dimension n. Let P(E) be the projective bundle of E, H the Hopf bundle over P(E) and $p: P(E) \longrightarrow X$ the natural projective. Define E' over P(E) by the relation :

$$p^*(E) = H \oplus E'$$

There is a natural identification : $F(E) = F(E')$ by sending the flag $0=M_0, M_1, M_2, \dots, M_n = E_x$ $(x \in X)$ of linear subspaces of E_x to the flag $M_0, M_2/M_1, M_3/M_1, \dots, M_n/M_1 = E'_{M_1}$. Under this identification, the natural line bundles $L_2, \dots, L_n$ over F(E) correspond to the natural line bundles $L_1', L_2', \dots, L'_{n-1}$ over $F(E')$, and L_1 over F(E) corresponds to H lifted to $F(E')$. Since $K_G^*(P(E))$ is freely generated over $K_G^*(X)$ by $1, H, \dots, H^{n-1}$ and the diagram :

$$\begin{tikzcd} F(E) \arrow[d] \arrow[r, equal] & F(E') \arrow[d] \\ X & P(E) \arrow[l] \end{tikzcd}$$

is commutative, the proposition is then true for $\dim E = n$.

By induction, it is true $\forall n$. Q.E.D.

Cor 3.3.2:

Let V be a G-module, L a 1 - diml. G-module. The natural map:

$$K_G^*(F(V \oplus L)) \xrightarrow{i^!} K_G^*(F(V))$$

is an epimorphism.

Proof:

Follows from the above proposition because each of the natural line bundles over $F(V)$ is the restriction of a natural one over $F(V \oplus L)$. Q.E.D.

Lemma 3.3.3:

Let V be a G-module, $G_k(V)$ the Grassmann manifold (of k-dim. subspaces) of V. Let $F(V)$ be the flag manifold of V. $\exists$ an embedding $K_G^*(G_k(V)) \xrightarrow{y} K_G^*(F(V))$ with a natural left inverse:

$$Y_!: \; K_G^*(F(V)) \longrightarrow K_G^*(G_k(V))$$

Proof:

Let $p(V)$ be the associated principal bundle. Because $U(k) \times U(n-k)$ acts freely on $p(V)$ $(n=|V|)$, then $K_G^*(G_k(V)) =$ $K_G^*(p(V)/U(k) \times U(n-k)) = K^*_{G \times U(k) \times U(n-k)}(p(V))$

([20]). Similarly if T is a maximal torus in $U(n)$, then $K_G^*(F(V)) = K_{G \times T}^*(p(V))$. It is shown by Atiyah ([2]) that the restriction homomorphism :

$$K^*_{G \times U(k) \times U(n-k)}(p(V)) \longrightarrow K^*_{G \times T}(p(V))$$

has a functional left inverst. Hence the result. Q.E.D.

Lemma 3.3.4:

Let V be a G-module, L a 1-dim. G-module. The G-space $G_k(V \oplus L)/G_k(V)$ is G-homeomorphic to the Thom space of a G-vector bundle (of a certain dimension) over $G_{k-1}(V)$. This is proved in the ordinary case, $G=e$, by S. Hoggar [13]. His proof is still valid in this general setting.

Proposition 3.3.5:

(i) If L is a 1 - dim. G-module, and V is a G-module, the natural homomorphism :-

$$K_G^*(G_k(V \oplus L)) \xrightarrow{j^!} K_G^*(G_k(V))$$

is epi.

(ii) Given a decomposable G-module V, then $K_G^*(G_k(V))$ is a finitely-generated free RG - module.

Proof:

(i) By Cor 3.3.2, the natural homomorphism $K_G^*(F(V \oplus L)) \xrightarrow{i^!} K_G^*(F(V))$. is epi. The diagram:

$$\begin{tikzcd} K_G^*(G_k(V \oplus L)) \arrow[r, "j^!"] & K_G^*(G_k(V)) \\ K_G^*(F(V\oplus L)) \arrow[u, "\gamma_!"] \arrow[r, "i^!"] & K_G^*(F(V)) \arrow[u, "\gamma_!"'] \end{tikzcd}$$

is commutative. Thus $j^! \circ \gamma_!$ is an epimorphism. This implies $j^!$ is epi.

(ii) Follows from (i) by using the Thom isomorphism theorem for K_G - theory ([2]), and Lemma 3.3.4. Q.E.D.

Proposition 3.3.6:

Let V be a G-module (G abelian). Then $U_G^*(G_k(V))[\Lambda^{-1}]$ is freely generated over $U_G^*[\Lambda^{-1}]$ by a finite number of elements $x_1, \ldots, x_l$ such that $\bar{\mu}(x_1), \bar{\mu}(x_2), \ldots, \bar{\mu}(x_l)$ generate $K_G^*(G_k(V))$ freely over RG.

Proof:

By induction on k. If $k = 1$, $G_k(V) = P(V)$ and the result follows from Proposition 2.

Suppose it is true for $k - 1$, and we want to prove it for $k (k \geq 2)$. We proceed by induction on $|V|$. For $|V| = k$, $G_k(V) = pt.$ and the result is true. Suppose the proposition has been proved for $|V| = n$. Let L be a 1 - dim. G-module.

We have a commutative diagram:

$$\begin{tikzcd} U_G^*(G_k(V\oplus L), G_k(V))[\Lambda^{-1}] \arrow[r] \arrow[d, "\bar{\mu}"] & U_G^*(G_k(V\oplus L))[\Lambda^{-1}] \arrow[r, "i^*"] \arrow[d, "\bar{\mu}"] & U_G^*(G_k(V))[\Lambda^{-1}] \arrow[d, "\bar{\mu}"] \\ K_G^*(G_k(V\oplus L), G_k(V)) \arrow[r] & K_G^*(G_k(V\oplus L)) \arrow[r, "i^!"] & K_G^*(G_k(V)) \end{tikzcd}$$.

By hypothesis, $U_G^*(G_k(V)$ is freely generated by $x_1, \ldots, x_l$ such that $y_1, \ldots, y_l$ ($y_i = \bar{\mu}(x_i)$) generate $K_G^*(G_k(V))$ freely over RG. By Proposition 3.3.5 (i), y_i can be pulled back to an element $y_i' \in K_G^*(G_k(V \oplus L))$. Since $\bar{\mu}: U_G^*(G_k(V \oplus L))[\Lambda^{-1}] \longrightarrow K_G^*(G_k(V \oplus L))$ is epi (Cor 3.2.2), y_i' can be pulled back to some element in $U_G^*(G_k(V \oplus L))[\Lambda^{-1}]$.
In particular, $y_1' = \bar{\mu}(t)$, some t. Let $z_1 = i^*(t)$. Then $\bar{\mu}(z_1) = \bar{\mu} i^*(t) = i^! \bar{\mu}(t) = y_1$. By the induction hypothesis, $\exists\, a_1, \ldots, a_l \in U_G^*[\Lambda^{-1}]$ such that $z_1 = \sum a_i x_i$.
Hence $\bar{\mu}(z_1) = \sum \bar{\mu}(a_i) y_i$ i.e. $y_1 = \sum \bar{\mu}(a_i) y_i$. This implies $\bar{\mu}(a_1) = 1$. But $a_1 = [{}^{S}/_{S'}]$ where $\bar{\mu}(s') = 1$.
Therefore, $\bar{\mu}(s) = 1$ i.e. s is invertible in $U_G^*[\Lambda^{-1}]$, and so is a_1. Hence if we replace x_1 by z_1, we still have a basis $z_1, x_2, x_3, \ldots, x_l$: of $U_G^*(G_k(V)[\Lambda^{-1}]$.
Proceeding in this way, we obtain a basis $z_1, \ldots, z_l$ of $U_G^*(G_k(V))[\Lambda^{-1}]$ such that $z_j = i^*(t_j)\ \forall j$.
Hence $i^*: U_G^*(G_k(V \oplus L)[\Lambda^{-1}] \longrightarrow U_G^*(G_k(V))[\Lambda^{-1}]$ is epi.
Since the latter module is free, the sequence:

$$U_G^*(G_k(V \oplus L), G_k(V))[\Lambda^{-1}] \to U_G^*(G_k(V \oplus L))[\Lambda^{-1}] \xrightarrow{i^*} U_G^*(G_k(V))[\Lambda^{-1}]$$

is a split s.e.s.
Now using the Thom isomorphism theorem (Th 1.4.5), Lemma 3.3.4, Lemma 2.1.7, and the induction hypothesis, we see that the result is valid for $G_k(V \oplus L)$. But G is abelian and so every G-module is decomposable. Hence the result. Q.E.D.

§ 3.4 : Equivariant K-theory and Cobordism (G abelian):

We are now in a position to relate K_G - theory to U_G^* - theory for G abelian. The non-abelian case is also true but needs more work. The next chapter will be concerned mainly with that.

Regard RG as a $U_G^*[\Lambda^{-1}]$ - module via the homomorphism: $\bar{\mu} : U_G^*(\Lambda^{-1}) \longrightarrow RG$ ($K_G^1(pt) = 0$). Given (X,A) in ($\underline{c}_1$), define :

$L_G^0(X,A) = U_G^{ev}(X,A)[\Lambda^{-1}] \otimes_{U_G^*[\Lambda^{-1}]} RG$, $L_G^1(X,A) = U_G^{od}(X,A)[\Lambda^{-1}] \otimes_{U_G^*[\Lambda^{-1}]} RG$

and $L_G^*(X,A) = L_G^{0q}(X,A) \oplus L_G^1(X,A)$ ... (3.4.1)

There are natural homomorphisms (i) $\beta : U_G^*(X,A)[\Lambda^{-1}] \to L_G^*(X,A)$ defined by $\beta(x) = x \otimes 1$, and (ii)

$$\hat{\mu} : U_G^*(X,A)[\Lambda^{-1}] \otimes_{U_G^*[\Lambda^{-1}]} RG \longrightarrow K_G^*(X,A)$$

with $\hat{\mu}(x \otimes y) = y \cdot \bar{\mu}(x)$. Hence there is a commutative diagram:

$$\begin{tikzcd} U_G^*(X,A)[\Lambda^{-1}] \arrow[rr, "\beta"] \arrow[dr, "\bar{\mu}"'] & & L_G^*(X,A) \arrow[dl, "\hat{\mu}"] \\ & K_G^*(X,A) & \end{tikzcd}$$

Define $\hat{c}_0 : K_G^*(X,A) \longrightarrow L_G^*(X,A)$ by the composition:

$$K_G^*(X,A) \xrightarrow{c_0} U_G^*(X,A)[\Lambda^{-1}] \xrightarrow{\beta} L_G^*(X,A)$$

Then $\hat{\mu} \circ \hat{c}_0 = -\,\mathrm{id} : K_G^*(X,A) \longrightarrow K_G^*(X,A)$. Hende $\hat{\mu} : L_G^*(X,A) \longrightarrow K_G^*(X,A)$ is epi.

We can define, too, a U_G^* - module structure on RG by means of $\tilde{\mu}: U_G^* \longrightarrow RG$. In a similar way, $U_G^*(X,A) \otimes_{U_G^*} RG$ is $\mathbb{Z}_2$ - graded via:

$$U_G^*(X,A) \otimes_{U_G^*} RG = U_G^{ev}(X,A) \otimes_{U_G^*} RG \oplus U_G^{od}(X,A) \otimes_{U_G^*} RG,$$

<u>Th 3.4.2</u> :

Let G be abelian. Suppose X is a locally compact G-space, and A a closed G-subspace of X. The homomorphism:

$$\mu \otimes \mathrm{id} : U_G^*(X,A) \otimes_{U_G^*} RG \longrightarrow K_G^*(X,A)$$

is an isomorphism of $\mathbb{Z}_2$ - graded rings.

<u>Proof</u>:

Because of the identification $U_G^*(X,A) \otimes_{U_G^*} RG = U_G^*(X,A) \otimes_{U_G^*} (U_G^*[\Lambda^{-1}] \otimes_{U_G^*[\Lambda^{-1}]} RG)$

$$= (U_G^*(X,A) \otimes_{U_G^*} U_G^*[\Lambda^{-1}]) \otimes_{U_G^*[\Lambda^{-1}]} RG = U_G^*(X,A)[\Lambda^{-1}] \otimes_{U_G^*(\Lambda-1)} RG,$$

it is enough to show that

$$\hat{\mu}: L_G^*(X,A) \longrightarrow K_G^*(X,A)$$

is an isomorphism of $\mathbb{Z}_2$ - graded rings. We have already shown it is an epimorphism, and so it only remains to prove that it is a monomorphism. Let V be a G-module, $G_k(V)$ the Grassmann manifold of k-dim. Subspaces of V and $M_k(V)$ the Thom space of the standard bundle over $G_k(V)$. It follows from Proposition 3.3.6 that $\hat{\mu}: L_G^*(G_k(V)) \xrightarrow{\cong} K_G^*(G_k(V))$ (See [8] P.60-61).

By the Thom isomorphism theorem for $U_G^*[\Lambda^{-1}]$ - theory, and the commutative diagram (Lemma 2.1.7):

$$\begin{tikzcd} L_G^*(G_k(V)) \arrow[r, "\varphi"] \arrow[d, "\hat{\mu}"'] & L_G^*(M_k(V)) \arrow[d, "\hat{\mu}"] \\ K_G^*(G_k(V)) \arrow[r, "\varphi", "\cong"'] & K_G^*(M_k(V)) \end{tikzcd}$$

$\hat{\mu}: L_G^*(M_k(V)) \xrightarrow{\cong} K_G^*(M_k(V))$. This implies the general case by using essentially the argument of Conner-Floyd [8] Theorem (10.1). Q.E.D.

Chapter 4 : The Relation of K_G - theory to Cobordism:

We wish to prove Th 3.4.2 for general G. This is achieved by studying the relation of $U_{U(n)}^*(-)$ and $U_T^*(-)$ where T is a maximal tonus of U_n.

§ 4.1 : The Relation of $U_G^*(-)$ with Bordism:

Closely related to U_G^* is the bordism ring $\mathcal{U}_G^*$ of unitary G-manifolds (for the definition see T. tom Dieck [11] P.29). Recall that an element of $\mathcal{U}_G^{-n}$ ($n \in \mathbb{Z}^+$) is represented by a closed unitary G-manifold ([11] P.29) N^n of dimension n, and $[N_1^n] + [N_2^n] = [N_1^n \cup N_2^n]$. Multiplication in $\mathcal{U}_G^* = \sum \mathcal{U}_G^{-n}$ is induced by cartesian product : $[N_1^k] \times [N_2^l] = [N_1^k \times N_2^l]$.

$\mathcal{U}_G^*$ has a unit $1 \in \mathcal{U}_G^0$ represented by the G-manifold consisting of a single point.

In the manner of Thom [23], one can define a ring homomorphism (T. tom Dieck [11]).

$$\iota : \mathcal{U}_G^* \longrightarrow U_G^* = U_G^*(pt),$$

as follows. Suppose $x \in \mathcal{U}_G^{-n}$ is represented by the closed unitary G-manifold M^n, n even, Embed M in a G-module V (Palais [18]) such that the normal bundle N of M in V receives the correct complex structure as a G-vector bundle (by the definition of unitary G-manifold [11]). Now N can be identified with a G-tubular neighbourhood of M in V. Let T, ∂T correspond to DN, SN, the disc bundle and sphere bundle of N respectively. Define $\iota(x) \in U_G^{-n}$ to be the element represented

$$V^+ \xrightarrow{(1)} V^+/(V^+ \setminus \text{int } T) \cong T/_{\partial T} \cong DN/_{SN} \xrightarrow{(2)} M_{|V| - n/2}(G)$$

where (1) is the collapsing map (Thom construction) and (2) is induced by the G-unitary structure on M. If n is odd, embed in $V \oplus \mathbb{R}$ ($\mathbb{R}$ the reals with trivial G-action) and carry out the same construction.

Lemma 4.1.1:

$$\iota : \mathcal{U}_G^* \longrightarrow U_G^*$$

is well-defined.

Proof:

(i) ι does not depend on the choice of V:

Suppose $f : M \longrightarrow V$, $f' : M \longrightarrow V'$ are two permissible embeddings of M. There is the diagonal embedding:

d: $M \longrightarrow V \oplus V'$ given by $d(m) = f(m) \oplus f'(m)$.

We need only show that f and d give the same answer for $i([M])$. Consider the G-homotopy of embeddings $M \to V \oplus V'$ defined by $d_t(m) = f(m) + t f'(m)$ $(0 \leq t \leq 1)$. This is a G-homotopy between d_0 and $d_1 = d$. Since i depends only on the G-homotopy class, it is enough to compare $f: M \to V$ and $d_0 : M \to V \oplus V'$. Let N be the normal bundle of f (M). Then $N \oplus V'$ is the normal bundle of d_0 (M) in $V \oplus V'$. Hence it is easily seen that the representative of $i(x)$ given by considering the embedding $d_0 : M \to V \oplus V'$ is given by suspending the G-map representing $i(x)$, (using $f : M \to V$), by V'^{+}. This completes the proof of (i).

(ii) i does not depend on the representative of the bordism class : see C - F [7] P.28 and tom Dieck [11] P33 . Q.E.D.

It has been shown (Thom-Milnor [23] and [17]) that in the ordinary case $i : \mathfrak{U}^* = \mathcal{N}_e^* \longrightarrow U^*$ is an isomorphism.
The inverse homomorphism is defined by using:

<u>Th 4.1.2:</u>

Given a base point preserving map:

$$f: S^{n+k} \longrightarrow M_k ,$$

$\exists$ a based map : $h : S^{n+k} \longrightarrow M_k$ s.t.

(i) h is homotopic to f , the homotopy fixed at the base points,

(ii) h is differentiable and (iii) h is transverse regular to $B_k = B_k(e)$

(Proved as in [15]).

Remark 4.1.3:

For G abelian, $G \neq e$ the map:

$$i : \mathcal{U}_G^* \longrightarrow U_G^*$$

is not an isomorphism.

Proof:

Let V be a non-trivial irreducible representation of G. Thus it is one-dimensional. The space $M_1(G)$ can be naturally identified with $P(V^\infty \oplus 1)$ = the projective space of $V^\infty \oplus 1$ (see Lemma 2.1.3) and so we can define a based G-map :

$$P(V)^+ \xrightarrow{f} M_1(G)$$

by sending $P(V) = \text{point} \longrightarrow V \in M_1(G)$. It is unique up to G-homotopy (Proposition 1.3.1). Let $\rho \in U_G^2(\text{point})$ be the element represented by this map. By Cor 2.1.6 , λ^{-1} is represented by $1 - H \in \tilde{K}_G(P(W \oplus 1))$ $\forall$ G-module W, H being the Hopf bundle over $P(W \oplus 1)$. It follows that $\mu(\rho) =$ $= 1 - V \neq 0$ in $K_G(pt) = RG$, since the latter is the free group generated by the irreducible, inequivalent representations of G. Hence U_G^2 has non-zero elements e.g. ρ. But since $\mathcal{U}_G^n = 0$ for all $n > 0$, then $i : \mathcal{U}_G^* \longrightarrow U_G^*$ is not an isomorphism.

Q.E.D.

Remark 4.1.4:

The equivariant analogue of Th 4.1.2 is false (at least for G abelian, $G \neq e$) .

Proof:

Suppose otherwise. Let $P(V)^{+} \xrightarrow{f} M_1(G)$ be the G-map we have defined in the proof of Remark 4.1.3. Then $\exists$ a G-module W such that $P(V)^{+} \xrightarrow{f} M_1(G)$ can be factored in the form

$P(V)^{+} \xrightarrow{j} P(W \oplus 1) \subset M_1(G)$ and j is transverse regular on $P(W) \subset P(W \oplus 1)$. This emplies that j assigns to $P(V)$ a point of $P(W \oplus 1) - P(W)$. The latter G-deformation retracts onto $P(1)$. Hence f is G-homotopic to the map sending $P(V)^{+}$ to the base point of $M_1(G)$ (by a based homotopy). Hence the element $\rho \in U_G^2$ represented by f is zero - contradiction (see the previous proof). Q.E.D.

§4.2 : A Gysin homomorphism:

In this section we assume that G is a compact connected Lie group. Let T be a maximal tonus of G. ${}^{G}/_{T}$ is a complex G-manifold which admits a complex G embedding in a G-module V. In the usual way, the normal bundle of ${}^{G}/_{T}$ in V can be identified with a tubular neighbourhood N of it. Let $V^{+} \xrightarrow{f} N^{+}$ be the collapsing map (Thom construction) and let $N^{+} \xrightarrow{h} M_k(G)$ be induced by a classifying map of the complex G-vector bundle $N \longrightarrow {}^{G}/_{T}$. As pointed out in §4.1, the composition : $V^{+} \xrightarrow{f} N^{+} \xrightarrow{h} M_k(G)$ is a representative of $i([{}^{G}/_{T}])$ where $i : \mathcal{U}_G^{*} \longrightarrow U_G^{*}$ is the natural map. Suppose now X is a compact G-space. Let

$$\Phi : U_G^*(G/T \times X) \longrightarrow \tilde{U}_G^*(N^+ \wedge X^+),\ \varphi' : U_G^*(X) \longrightarrow \tilde{U}_G^*(V^+ \wedge X^+)$$

be the Thom isomorphisms (Th 1.4.5). Define

$$P_* : U_G^*(G/T \times X) \longrightarrow U_G^*(X)$$

to be equal to the composition:

$$U_G^*(G/T \times X) \xrightarrow{\varphi} \tilde{U}_G^*(N^+ \wedge X^+) \xrightarrow{f^* \wedge 1} \tilde{U}_G^*(V^+ \wedge X^+) \xrightarrow{\varphi'^{-1}} U_G^*(X)\ .$$

Lemma 4.2.1:

P_* is a well-defined homomorphism.

Proof:

One needs to verify that P_* does not depend on the choice of the embedding of G/T. We use an Atiyah type of argument (cf [5] 498-499). As in the proof of Lemma 4.1.1, it is enough to compare the results of embeddings $S: G/T \longrightarrow V$ and $d_o : G/T \longrightarrow V \oplus V'$ such that if N is the normal bundle of S in V, then $N \oplus V'$ is the normal bundle of d_o in $V \oplus V'$.

Denote by S_* the composition

$$U_G^*(G/T \times X) \xrightarrow{\varphi} \tilde{U}_G^*(N^+ \wedge X^+) \xrightarrow{f^* \wedge 1} \tilde{U}_G^*(V^+ \wedge X^+)$$

and by d_{o_*} the composition

$$U_G^*(G/T \times X) \xrightarrow{\varphi} \tilde{U}_G^*((N \oplus V')^+ \wedge X^+) \xrightarrow{f^* \wedge 1} \tilde{U}_G^*((V \oplus V')^+ \wedge X^+).$$

Because of the transitivity of the Thom homomorphism, the diagram

$$\begin{tikzcd}
 & U_G^*(G/T \times X) \arrow[dl, "S_*"'] \arrow[dr, "d_{o_*}"] & \\
\tilde{U}_G^*(V^+ \wedge X^+) \arrow[rr, "\varphi"] & & \tilde{U}_G^*((V \oplus V')^+ \wedge X^+) \\
 & U_G^*(X) \arrow[ul, "\varphi"] \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

is commutative. Hence the lemma. Q.E.D.

Let $p^* : U_G^*(X) \longrightarrow U_G^*({}^G/_T \times X)$ be the natural homomorphism induced by the projection ${}^G/_T \times X \longrightarrow X$.

Proposition 4.2.2:

Given a compact G-space X, then for all $x \in U_G^*({}^G/_T \times X)$, and $y \in U_G^*(X)$, $p_* (x \cdot p^*(y)) = p_*(x) . y$. Moreover the composition $U_G^*(X) \xrightarrow{p^*} U_G^*({}^G/_T \times X) \xrightarrow{p_*} U_G^*(X)$ is multiplication by the G-bordism class $[{}^G/_T] \in \mathcal{U}_G^*$.

Proof:

The first part follows by the method of Dyer [12] P54. Since X is compact, $U_G^*({}^G/_T \times X)$ has an identity 1, so that $p_* \circ p^*(y) = p_*(1)\, y$. The naturality of p_* gives rise to a commutative diagram:

$$\begin{tikzcd} U_G^*({}^G/_T) \arrow[r, "p_*"] \arrow[d, "k^*"'] & U_G^* \arrow[d, "k^*"] \\ U_G^*({}^G/_T \times X) \arrow[r, "p_*"] & U_G^*(X) \end{tikzcd}$$

where k^* is induced by $X \longrightarrow$ point. Hence $p_*(1) = k^*(p'_*(1))$.
Going back to the first paragraph of this section, we see that $p'(1)$ is represented by the composition $V^+ \xrightarrow{f} N^+ \xrightarrow{h} M_k(G)$
i.e. $p'_*(1) = ([{}^G/_T])$. Q.E.D.

By Lemma 1.4.7. $p^* : U_G^*(X) \longrightarrow U_G^*({}^G/_T \times X)$ can be factored as the composition:

$$U_G^*(X) \xrightarrow{\Gamma} U_T^*(X) \xrightarrow{F(T,G)} U_G^*(G \times_T X) \xrightarrow{v^{*-1}} U_G^*({}^G/_T \times X) .$$

Define $r_! : U_T^*(X) \longrightarrow U_G^*(X)$ by the composition:

$$U_T^*(X) \xrightarrow{F(T,G)} U_G^*(G \underset{T}{\times} X) \xrightarrow{v^{*-1}} U_G^*({}^G/_T \times X) \xrightarrow{P_*} U_G^*(X).$$

We summarize the results of this section in

Cor 4.2.3:

Let $r : U_G^*(X) \longrightarrow U_T^*(X)$ be the restriction homomorphism (X compact). There is a natural homomorphism

$$r_! : U_T^*(X) \longrightarrow U_G^*(X)$$

such that $r_! \circ r$ is multiplication by $[{}^G/_T]$.

4.3 : The Equivariant Todd Genus of ${}^G/_T$:

We assume the following from [4] sections 6.2, 6.3 (see also [5]).

Th 4.3.1:

Let $\tau{}^G/_T$ be the tangent bundle of ${}^G/_T$, $\varphi : K_G({}^G/_T) \longrightarrow K_G(\tau\, {}^G/_T)$ the Thom homomorphism, and t-ind : $K_G(\tau\, {}^G/_T) \longrightarrow RG$ the topological index. Then t-ind $(\varphi(1)) = 1$.

This implies

Proposition 4.3.2:

The composition $\mathcal{U}_G^* \xrightarrow{i} U_G^* \xrightarrow{\mu} RG$ assigns to $[{}^G/_T]$ the identity of RG.

Proof:

There is a complex embedding of ${}^G/_T$ in a G-module V. In the usual way, the normal bundle N to ${}^G/_T$ in V can be identified with a G-tubular neighbourhood of it. There is then induced a complex embedding of ${}^G/_T$ in τV with normal bundle τN. Let $(\tau V)^+ \xrightarrow{k} (\tau N)^+$ be the collapsing map and let $(\tau N) \xrightarrow{+f} M_r(G)$

be induced by a classifying map of $\tau N \to G/T$ $(r = 2|V| - |G/T|)$. By the definition of $i : \mathcal{N}^*_G \to U^*_G$, $i([G/T])$ is represented by the composition:

$(\tau V)^+ \xrightarrow{k} (\tau N)^+ \xrightarrow{f} M_r(G)$. Hence $\mu i([G/T])$ is the image of the natural element $\bar{\lambda}^r$ in $\tilde{O}(M_r(G))$ under the composition:

$$\tilde{O}(M_r(G)) \xrightarrow{f^!} \tilde{K}_G((\tau N)^+) \xrightarrow{k^!} \tilde{K}_G((\tau V)^+) \xrightarrow{\varphi^{-1}} RG.$$

By naturality of the Thom class, $f^!(\lambda^r)$ = the Thom class in $\tilde{K}_G((\tau N)^+)$ of the bundle $\tau N \to G/T$. Denoting the Thom homomorphism by φ, we deduce that applying the composition:

$$K_G(G/T) \xrightarrow{\varphi} K_G(\tau N) \xrightarrow{k^!} K_G(\tau V) \xrightarrow{\varphi^{-1}} RG$$

to the element $1 \in K_G(G/T)$ gives us $\mu i([G/T])$. Let E_x denote the fiber over the point x of the bundle $E \to X$. Given $x \in G/T$, $(\tau N)_x = N_x \oplus V = N_x \oplus N \oplus (\tau G/T)_x$. Hence $\tau G/T \to G/T$ is naturally a G-sub bundle of $\tau N \to G/T$ (when both are considered as complex bundles). By the transitivity of the Thom homomorphism ([20]), we get a commutative diagram:

$$\begin{tikzcd}
K_G(G/T) \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & K_G(\tau N) \arrow[r, "k^!"] & K_G(\tau V) \arrow[r, "\varphi^{-1}"] & RG \\
& K_G(\tau G/T) \arrow[u, "\varphi"] & &
\end{tikzcd}$$

Therefore, $\mu i([G/T]) = t\text{-ind}(\varphi(1)) = 1$ (Theorem 4.3.1).

Q.E.D.

Remark:

Conner-Floyd have proved that for ordinary unitary bordism, the composition $\mathcal{U}_* \xrightarrow{i} U^* \xrightarrow{\mu} \mathbb{Z}$ sends the bordism class of a manifold M^{2n} into $(-)^n \ Td[M^{2n}]$ where $Td[M^{2n}]$ is the Todd genus of M^{2n} ([8] P.37). So we view the above proposition as a computation (up to sign) of the equivariant Todd genus of ${}^G/_T$.

§4.4: The Main Theorem:

Proceeding as in § 4.2, we can construct a homomorphism

$$P_! \ : K_G^*\left({}^G/_T \times X\right) \longrightarrow K_G^*(X)$$

for all compact G-spaces X (G connected). Let $r' : K_G^*(X) \longrightarrow K_T^*(X)$ be the restriction homomorphism, and let $K_T^*(X) \xrightarrow{F'(T,G)} K_G^*(G \times_T X)$ be the canonical isomorphism ([20] P.132). Using the identification $v: G \times_T X = {}^G/_T \times X$ (1.4.6), we get an isomorphism

$$v_! : K_G^*\left({}^G/_T \times X\right) \cong K_G^*(G \times_T X) \ .$$

Lemma 4.4.1:

The diagrams:

(i)
$$\begin{tikzcd} U_G^*(X) \arrow[r, "r"] \arrow[d, "\mu"'] & U_T^*(X) \arrow[d, "\mu"] \\ K_G^*(X) \arrow[r, "r'"] & K_T^*(X) \end{tikzcd}$$

(ii)
$$\begin{tikzcd} U_T^*(X) \arrow[r, "F(T,G)"] \arrow[d, "\mu"'] & U_G^*(G \times_T X) \arrow[d, "\mu"] \\ K_T^*(X) \arrow[r, "F'(T,G)"] & K_G^*(G \times_T X) \end{tikzcd}$$

(iii)
$$\begin{tikzcd} U_G^*\left({}^G/_T \times X\right) \arrow[r, "P_*"] \arrow[d, "\mu"'] & U_G^*(X) \arrow[d, "\mu"] \\ K_G^*\left({}^G/_T \times X\right) \arrow[r, "P_!"] & K_G^*(X) \end{tikzcd}$$

are commutative.

Proof:

The proof of (i) is straightforward. To prove (ii) is commutative, we recall that the inverse isomorphisms

$F(G,T) : U^*_G(G\times_T X) \longrightarrow U^*_T(X)$ and $F'(G,T) : K^*_G(G\times_T X) \longrightarrow K^*_T(X)$

are given by the compositions $U^*_G(G\times_T X) \xrightarrow{r} U^*_T(G\times_T X) \xrightarrow{q^*} U^*_T(X)$

and $K^*_G(G\times_T X) \xrightarrow{r} K^*_T(G\times_T X) \xrightarrow{q^*} K^*_T(X)$ where r is the restriction homomorphism and $q : X \to G\times_T X$ sends x to $[1, x]$.

Hence by (i) and the naturality of μ, (ii) is commutative.

Because μ is natural and commutes with the Thom homomorphism (Lemma 2.1.7), (iii) is commutative. Q.E.D.

Define $r_! : K^*_T(X) \longrightarrow K^*_G(X)$ by the composition

$$K^*_T(X) \xrightarrow{F(T,G)} K^*_G(G\times_T X) \xrightarrow{v_!^{-1}} K^*_G(G/_T \times X) \xrightarrow{P_!} K^*_G(X).$$

Hence $r_! \circ r' = \mathrm{id} : K^*_G(X) \longrightarrow K^*_G(X)$ (Atiyah [2]). On the other hand, the composition

(4.4.2) $$U^*_G(X) \otimes_{U^*_G} RG \xrightarrow{r \otimes r'} U^*_T(X) \otimes_{U^*_T} RT \xrightarrow{r_! \otimes r'_!} U^*_G(X) \otimes_{U^*_G} RG$$

is multiplication by $\mu([G/_T]) = 1$ (Cor 4.2.3 and Prop.4.3.2) i.e. the composition (4.4.2) is equal to id:

$$U^*_G(X) \otimes_{U^*_G} RG \longrightarrow U^*_G(X) \otimes_{U^*_G} RG .$$

Combining this result with Th. 3.4.2 and Lemma 4.4.1.

Theorem 4.4.3:

Let G be a compact connected Lie goup. The homomorphism

$$\mu' \otimes 1 \; : \; U^*_G(X) \otimes_{U^*_G} RG \longrightarrow K^*_G(X)$$

is an isomorphism of $\mathbb{Z}_2$ - graded rings for all compact G-spaces X. Moreover for any compact Lie group G, and a compact G-space X,

$$\mu'\otimes 1 : U^*_G(X) \otimes_{U^*_G} RG \cong K^*_G(X) .$$

<u>Proof</u>:

Case (i) : G connected. Let T be a maximal tonus of G.

By Theorem 3.4.2 , $\mu\otimes 1 : U^*_T(X) \otimes_{U^*_T} RT \cong K^*_T(X)$. From Lemma 4.4.1, the diagram:

$$\begin{tikzcd}
U^*_G(X)\otimes_{U^*_G} RG \arrow[r, "r\otimes r'"] \arrow[d, "\mu'\otimes 1"] & U^*_T(X)\otimes_{U^*_T} RT \arrow[r, "r_!\otimes r'_!"] \arrow[d, "\mu\otimes 1"] & U^*_G(X)\otimes_{U^*_G} RG \arrow[d, "\mu'\otimes 1"] \\
K^*_G(X) \arrow[r, "r'"] & K^*_T(X) \arrow[r, "r'_!"] & K^*_G(X)
\end{tikzcd}$$

is commutative. Since $\mu\otimes 1$ and $r\otimes r'$ are injective (4.4.2), then $\mu'\otimes 1$ is injective. It is also surjective because $r'_!$ and $\mu\otimes 1$ are. Hence it is an isomorphism.

Case (ii) : The general case. Embed G in U(n) for some n.

Then $U^*_G(X)\otimes_{U^*_G} RG \cong U^*_{U(n)}(U_n \times_G X) \otimes_{U^*_G} K^*_{U(n)}({}^{Un}/_G) \cong$

$\cong U^*_{U(n)}(U_n \times_G X) \otimes_{U^*_G} (U^*_{U(n)}({}^{U_n}/_G) \otimes_{U^*_n} RU_n)$

$\cong (U_{U(n)}(U_n \times_G X) \otimes_{U^*_G} U^*_G) \otimes_{U^*_n} RU_n \cong U^*_{U(n)}(U_n \times_G X) \otimes_{U^*_n} RU_n$

$\cong K^*_{U(n)}(U_n \times_G X) \cong K^*_G(X)$ where all the isomorphisms that appear are the canonical ones we already defined (see §1.4, and case (i) above).

REFERENCES

1. M.F. Atiyah , K-theory (Benjamin 1967)

2. On Both Periodicity, The Quarterly Journal of Maths Vol. 19, No. 74, June 1968

3. Bordism and Cobordism, Cambridge Phil.Soc. Vol.57, 1961

4. M.F. Atiyah, and G.B. Segal, K_G - theory Notes (Oxford) 1965

5. M.F. Atiyah and I.M. Singer, The index of Elliptic operators, Ann. of Math 87 (1968)

6. N. Bourbaki, 'Algebre commutative'. Ch. 2. Localization, Herman, Paris (1961)

7. P.E. Conner and E.E. Floyd, 'Differentiable Periodic Maps' Springer (1964)

8. , 'The relation of cobordism to K-theory' Lecture notes in Math. 28, Springer 1966

9. , 'Torsion in SU-bordism' Mem. Amer. Math. Soc. 60 (1966)

10. H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press (1956)

11. T. tom Dieck, 'Cobordism Theory and Transformation G maps' Aarhus Universitet (1968/69) No. 30

12. E. Dyer 'Cohomology Theories'. Benjamin (1969)

13. S.G.Hoggar 'On KO-theory of Grassmannians Ph.D thesis Warwick University

14. D. Husemoller 'Fibre Bundles' McGraw-Hill

15. J.W. Milnor 'Characteristic Classes' (1957)

16. 'Differential Topology Notes' Lect.Notes (1958)

17. 'On the cobordism ring....' Amer. J. Math. 82 (1960)

18. R.S. Palais, 'Immbedding of compact, diff. transormation groups', J. Math. Mech 6 (1957)

19. Pontjagin, Lie Groups

20. G.B. Segal, Equivariant K-theory, Publ. Math I. H.E.S. (1968)

21. , Classifying spaces, Publ. math I. H.E.S. (1968)

22. Spanier, Algebraie Topology

23. R. Thom, Quelques proprietes globales des varietes diff. Comment Math. Helv. 27 (1953)

24. G. Whitehead, 'Generalized homology theories' Trans. Amer. Math. Soc. 102, (1962)